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See page 5 for information on how to obtain the Answers.

2.3 Irrational Numbers

In practice, the only **irrational** numbers you will meet are surds and special constants like π. Surds include square roots (for example $\sqrt{2}$), cube roots (for example $\sqrt[3]{2}$), fourth roots (for example $\sqrt[4]{2}$), and so on. Their decimal patterns go on forever with no recurring pattern:

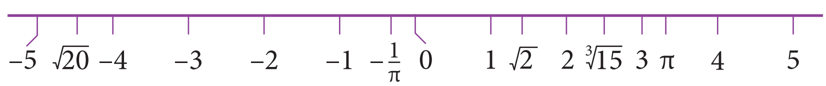
- $\sqrt{2} = 1.4142135623...$
- $\sqrt[3]{2} = 1.2599210498...$
- $\sqrt[4]{2} = 1.18920711...$

> **Note**: Remember that roots which evaluate to integers are **not** surds. That means $\sqrt{4} = 2$, $\sqrt[3]{27} = 3$ and $\sqrt[4]{625} = 5$ are not surds.

π is also an irrational number, including its powers and multiples. $\pi = 3.14159265358979323846...$

As π is irrational, its digits have no pattern. Calculating π to trillions of digits is a common test of power for a supercomputer.

The number line includes all real numbers. Both rational and irrational numbers are real numbers. The following number line shows a selection of both rational and irrational real numbers.



> **Note**: As the decimal pattern of an irrational number does not terminate, irrational numbers cannot be marked precisely on the real number line.

When an irrational number is multiplied or divided by a rational number, the result usually remains irrational. When a rational number is added to or subtracted from an irrational one, the result usually remains irrational. However, sometimes combining two irrational numbers **does** produce a rational number, as shown in the following two examples.

Example 5

Suggest values of p and q to show that two irrational numbers may add to give a rational one.

If $p = 2 + \sqrt{3}$ and $q = 2 - \sqrt{3}$

then $p + q = (2 + \sqrt{3}) + (2 - \sqrt{3}) = 4$ which is an integer and a rational number.

Example 6

Suggest values of x and y to show that two irrational numbers may multiply to give a rational one.

If $x = \sqrt{5}$ and $y = \sqrt{20}$

then $x \times y = \sqrt{5} \times \sqrt{20} = \sqrt{100} = 10$ which is an integer and a rational number.

You may be asked to state examples of irrational numbers.

Example 7

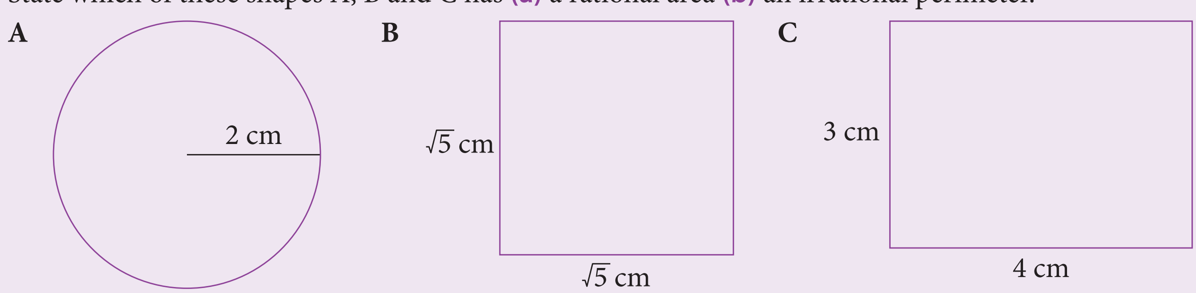
Write down an irrational number between 300 and 400

Possible answers:

- 100π (to use a multiple of π)
- The square root of any integer between $300^2 = 90\,000$ and $400^2 = 160\,000$ that is not a perfect square, such as $\sqrt{100\,000}$

Exercise 2C

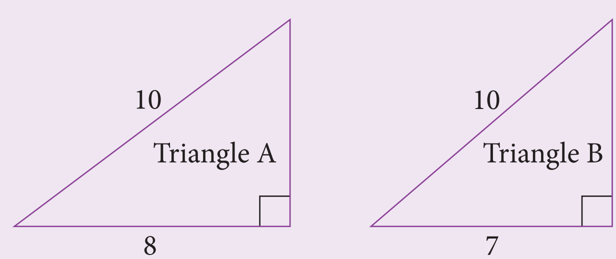
1. State whether each of the following numbers is rational or irrational:
   (a) $\sqrt{24}$ (b) $\sqrt{25}$ (c) $\sqrt[4]{26}$ (d) $\sqrt[3]{27}$
2. State whether each of these shapes has a rational area:
   (a) A circle of radius $\sqrt{7}$ (b) A square of side $\sqrt{7}$
3. A rectangle has sides $\sqrt{2}$ and $\sqrt{8}$. State whether each of these quantities is rational or irrational:
   (a) The area of the rectangle. (b) The perimeter of the rectangle.
4. State whether the solutions of these equations are integers, rational fractions or irrational numbers:
   (a) $\frac{5}{4}x^2 = 0.45$ (b) $\frac{3}{2}x^2 = 6$ (c) $\frac{5}{3}x^2 = 20$
5. State an irrational number that can be added to the number $1 + \pi$ to form a rational result.
6. The areas of three circles are given below. In each case (a) to (c) state whether the radius of the circle is rational or irrational.
   (a) 9 cm^2 (b) 9π cm^2 (c) $9\pi^2$ cm^2
7. Write down an irrational number between:
   (a) 1 and 2 (b) 3 and 4 (c) 4 and 5
8. State which of these shapes A, B and C has (a) a rational area (b) an irrational perimeter.



9. Identify all the irrational values in the following list:

   π^2 $\frac{\sqrt{27}}{\sqrt{2}}$ $\frac{\sqrt{27}}{\sqrt{3}}$ $\frac{\sqrt{27}}{\sqrt{4}}$ $\sqrt[3]{27}$

10. The diagram shows two right-angled triangles, A and B. Determine whether the area of each triangle is rational or irrational. Show your working.



2.4 Summary

In this chapter you have learnt how to:

- Write decimals with a recurring pattern as exact fractions.
- Identify irrational numbers.
- Distinguish irrational numbers from rational numbers in a variety of algebraic and geometric contexts.

Chapter 6
Non-Linear Graphs

6.1 Introduction

In the M7 module you studied non-linear graphs, in particular quadratic graphs, cubic graphs and reciprocal graphs. This chapter assumes and builds upon this knowledge.

Key words

- **Non-linear graph**: Any graph that is not a straight-line graph.
- **Exponential function**: An exponential function is a relationship with the form $y = k^x$ or $y = ak^x$ where a and k are constants (i.e. fixed numbers), and x and y are variables. An exponential function can represent real-world growth or decay. For example, it can be used to model compound interest, population change and radioactive decay.
- **Gradient**: A measure of the steepness of a line or curve.
- **Tangent**: A straight line that touches a curve at one point.

Before you start you should know how to:

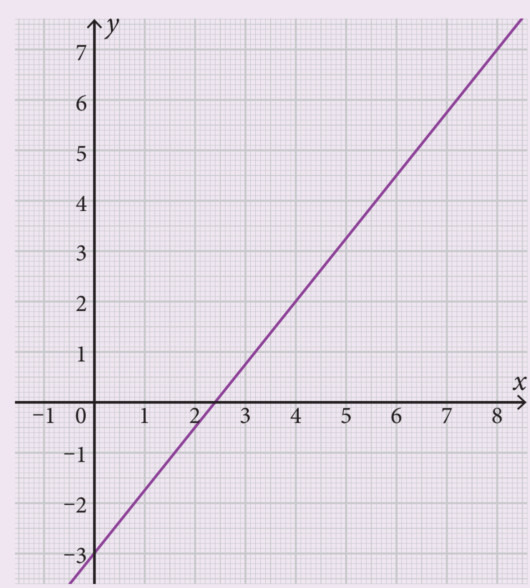
- Calculate the gradient of a straight-line graph.
- Recognise, sketch and interpret graphs of linear functions, quadratic functions, simple cubic functions and the reciprocal function.
- Interpret, order and calculate with numbers written in standard form.

In this chapter you will learn how to:

- Estimate the gradient of a curve at a point on the curve and how to interpret this gradient.
- Plot and use exponential graphs.
- Set up and solve equations relating to real-world examples of exponential growth and decay.

Exercise 6A (Revision)

1. Calculate the gradient of the straight line shown on the graph on the right.



2. A curve has the equation $y = x^2 - 2x + 3$
   (a) Copy and complete the following table of values for this curve.

x	−2	−1	0	1	2	3	4
y							

   (b) Plot the curve $y = x^2 - 2x + 3$, using values of x from −2 to 4
   (c) Write down the coordinates of the turning point of this curve.

3. On your calculator, work out the following. Give each answer in standard form, rounding to 4 significant figures where appropriate.
   (a) $3\times10^6 + 4\times10^5$
   (b) $(3.9\times10^6) \times (5.1\times10^{-4})$
   (c) $(2\times10^5) \times (1.03)^4$

6.2 Gradient of a Curve

You may be asked to find the gradient of a curve at a point and to interpret this gradient in context.

You can do this by drawing a **tangent** to the curve at the point. A tangent is a straight line that touches the curve at one point. You can work out the gradient of the tangent by finding two points (x_1, y_1) and (x_2, y_2) on the straight line and calculating $\frac{\text{rise}}{\text{run}}$ or using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$

The gradient of the curve at a point is equal to the gradient of the tangent.

The gradient at a point on a curve can be interpreted as the instantaneous rate of change.

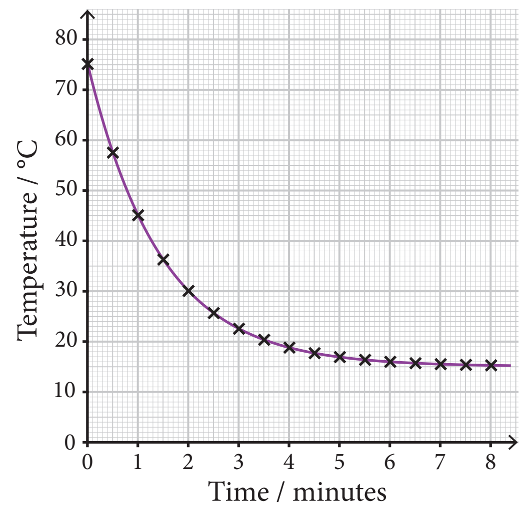
Example 1

The graph on the right shows the temperature of a cooling object plotted against time.

(a) Draw a tangent to the curve at the point where $t = 2$

(b) By working out the gradient of the tangent, estimate the gradient of the curve at this point.

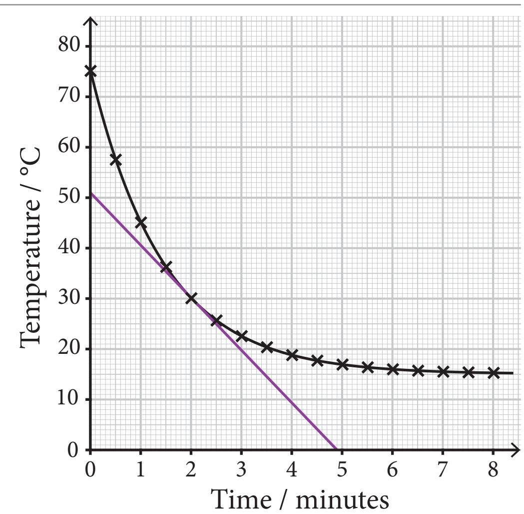
(c) Give a real-world interpretation of this gradient.



(a) The tangent is shown on the graph on the right.

(b) The tangent passes through the points (0, 51) and (4.9, 0). The gradient of the tangent is given by $m = \frac{51 - 0}{0 - 4.9} = -10.4$ to 3 significant figures. This is also the gradient of the curve when $t = 2$ minutes.

(c) The gradient of −10.4 means that the temperature is **decreasing** at a rate of 10.4 °C per minute at this time. The fact that the gradient is negative tells us that the temperature is decreasing at this rate. A positive gradient would indicate an increasing temperature.



Note: It is difficult to draw the tangent accurately by hand. This is why the gradient of the curve is **estimated** using this method. In an examination question, you will usually gain full marks if your method is correct and your calculation of the gradient is consistent with the tangent you have drawn.

9.3 The Cosine Rule

The cosine rule links all three side lengths in a triangle with one of the angles:

$a^2 = b^2 + c^2 - 2bc \cos A$

Notes: Following the standard notation, the angle A is always the angle opposite the side length a.

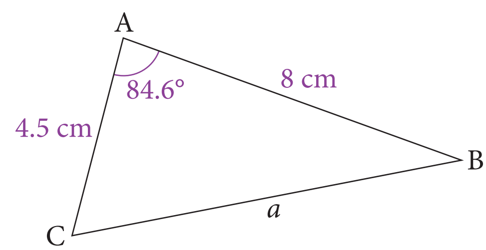
The cosine rule formula appears on the GCSE higher tier formula sheet, but it is a good idea to memorise this important formula.

In all questions involving the cosine rule, there is a lot of calculator work. It is a good idea to work to about 4 significant figures in your working. If you round the numbers in your working too much, you may lose accuracy in your final answer.

Note: The cosine rule is an extension of Pythagoras' Theorem for non-right-angled triangles, which is why it begins $a^2 = b^2 + c^2$

Example 6

Find the side length a in the triangle shown on the right, giving an answer to 1 decimal place.



$a^2 = b^2 + c^2 - 2bc \cos A$

$a^2 = 4.5^2 + 8^2 - 2(4.5)(8)\cos 84.6$

You can put the entire calculation into your calculator in one line:

$a^2 = 77.474...$

$a = 8.8$ cm (1 d.p.)

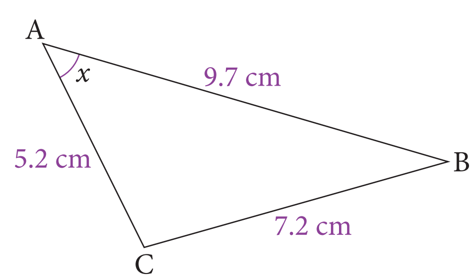
If all three side lengths are known, the cosine rule formula can be used to find an angle. In this case, this form of the formula may be used:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

This is demonstrated in Method 2 in the next example.

Example 7

Find the size of angle A, marked x in the triangle on the right.



Since the problem involves three side lengths and one angle, including the unknown, the cosine rule can be used.

Side length a must be opposite the angle we are finding, so $a = 7.2$

For the other two sides, we can choose which one to label b and which one c. Here we have taken $b = 5.2$ cm and $c = 9.7$ cm.

Method 1: use the formula on the formula sheet

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$7.2^2 = 5.2^2 + 9.7^2 - 2(5.2)(9.7)\cos x$$
$$51.84 = 121.13 - 100.88\cos x$$

> **Note**: $5.2^2 + 9.7^2$ and $2(5.2)(9.7)$ must be calculated separately, as in the line above.

Rearrange: $100.88\cos x = 121.13 - 51.8$

$$\cos x = 0.6868...$$
$$x = \cos^{-1}(0.6868...)$$
$$x = 46.6° \text{ (1 d.p.)}$$

Method 2: use the rearranged version of the formula

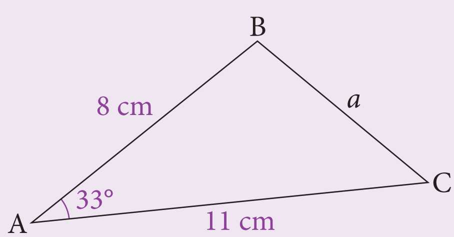
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos x = \frac{5.2^2 + 9.7^2 - 7.2^2}{2(5.2)(9.7)}$$
$$\cos x = 0.6868...$$
$$x = \cos^{-1}(0.6868...)$$
$$x = 46.6° \text{ (1 d.p.)}$$

> **Note**: Using the rearranged formula to find an unknown angle is quicker, but this formula does not appear on your formula sheet, so you must either memorise it or know how to derive it quickly.

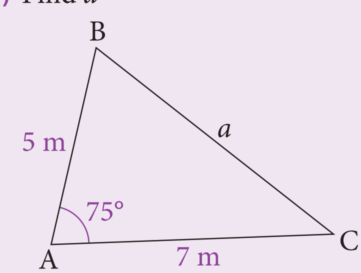
Exercise 9C

1. Find the lengths of the missing sides in the following triangles. Give your answers to 1 decimal place.

(a) Find a



(b) Find a

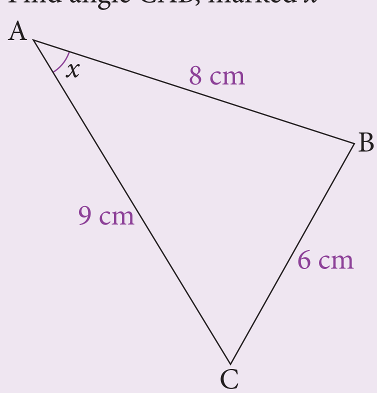


(c) In triangle ABC, AB = 6 km, BC = 2 km and angle ABC = 33°. Find the length of the side AC.

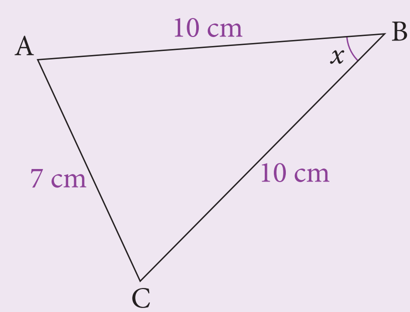
(d) In a triangle ABC, BC = 12 cm, AC = 9 cm and angle ACB = 64°. Find the length of the third side AB.

2. Find the size of the angle specified in these triangles, giving answers in degrees to one decimal place.

(a) Find angle CAB, marked x



(b) Find x



(c) In triangle PQR, QR = 9 m, PR = 11 m, PQ = 5 m. Find angle QPR.

(d) In triangle ABC, BC = 4 mm, AC = 5 mm, AB = 6 mm. Find angle ACB.

Chapter 11
Enlargement and Similarity

11.1 Introduction

In this chapter you will learn how to do an enlargement of a shape using a negative scale factor. You will also learn about the relationship between the surface areas and volumes of similar 3D shapes.

Key words

- **Transformation**: In GCSE mathematics you study four types of transformation: rotation, reflection, translation and enlargement.
- **Enlargement**: Changes the size of a shape without changing its shape.
- **Image**: The shape that results from a transformation.
- **Similar**: Shapes that are similar have the same shape but different sizes. If one shape is an enlargement of another, the two shapes are similar.

Before you start you should:

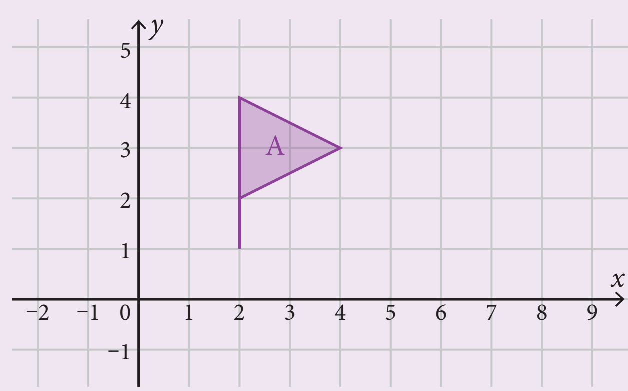
- Know how to enlarge a shape using a positive scale factor.
- Know that similar shapes (2D or 3D) are enlargements of each other.
- Know how to use the relationship between the ratios of lengths and areas of similar 2D shapes.
- Know how to use the effect of enlargement on the volume of a solid.

In this chapter you will learn about:

- A relationship between the ratios of lengths, areas and volumes of similar 3D shapes.
- A relationship between the surface areas of similar 3D shapes and between volumes of similar 3D shapes.

Exercise 11A (Revision)

1. Copy shape A on the right onto coordinate axes.
   (a) Enlarge shape A using a scale factor of 2 and a centre of enlargement (–1, 3). Label the image shape B.
   (b) Enlarge Shape A using a scale factor of $\frac{1}{2}$ and a centre of enlargement (–2, 3). Label the image shape C.



2. A company logo has a perimeter of 40 cm and an area of 190 cm^2. The logo is enlarged using a scale factor of 2 to create a similar image. Find:
   (a) The perimeter of the enlarged logo.
   (b) The area of the enlarged logo.
3. James makes a scale model of the Titanic Museum in Belfast. His model is 10 cm high and has a volume of 150 cm^3. Abi makes a similar model that is 20 cm high. Copy the following sentences, filling in the blanks to find the volume of Abi's model:

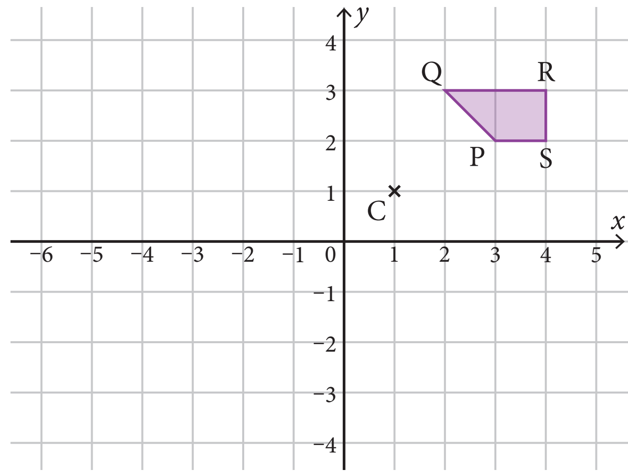
Abi's model is 2 times taller than James's, so the length scale factor is ______
The volume scale factor is 2^3, or ______
To calculate the volume of Abi's model, multiply the volume of James's model by ______ :
$8 \times$ ______ = ______ cm^3

11.2 Enlargement of 2D Shapes Using a Negative Scale Factor

In this section you will learn how to enlarge a 2D shape using a negative scale factor and a centre of enlargement.

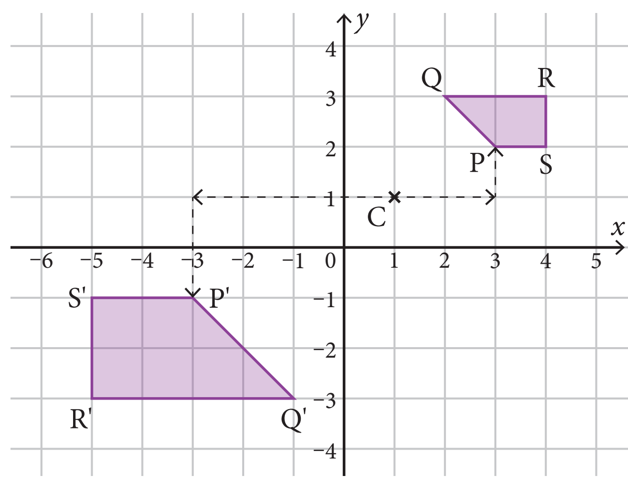
Example 1

Enlarge the shape PQRS, shown on the right, using a scale factor of –2 and a centre of enlargement C(1, 1). Label the image P'Q'R'S'.



For each vertex of the original shape:

- Calculate the vector from the centre of enlargement to the vertex. For example, from the centre C(1, 1) to P(3, 2), we move 2 to the right and 1 up, giving a vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
- Multiply both numbers in the vector by the scale factor. In this case, the scale factor is –2, giving $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$
- Use this vector to move from the centre of enlargement to the image point. Moving from (1, 1) using the vector $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$ takes us to P'(–3, –1), as shown in the diagram on the right.
- Repeat this for each vertex to get the positions of all four vertices in the image.
- Join the vertices in the image to form the enlarged shape, as shown in the diagram on the right.



Note: After an enlargement using a negative scale factor, the image has the same shape as the original, but it does not have the same orientation.

Describing an enlargement with a negative scale factor

If the two shapes are drawn on a grid with coordinate axes shown, you can describe the enlargement by giving the scale factor and the centre of enlargement.