Contents

See page 5 for information on how to obtain the Answers.

Chapter 2
Factorising Quadratic Expressions

2.1 Introduction

This chapter extends the factorisation that you studied at M3 to include every type of quadratic expression.

Key words

- **Linear expression**: A collection of terms, each either a constant or a variable to the power one, combined by addition or subtraction, for example: $3y + 4x - 8$ or $7x - 5 + 2x + 9$
- **Quadratic expression**: A group of terms whose highest power of any variable is two

Before you start you should know how to:

- How to factorise linear expressions such as $5x - 25y + 125$
- How to factorise quadratic expressions of the form $x^2 + bx + c$

In this chapter you will learn how to

- Factorise quadratic expressions of the form $ax^2 + bx + c$
- Factorise more complex expressions, for example $3x^2 - 75$, $x^2 + xy - 6y^2$ and $2px - qx - 2py + qy$

Example 1

Factorise $x^2 - 7x - 30$

Begin by seeking two numbers that add to give –7 and multiply to give –30
After a few tries, you should come up with +3 and –10 since:
$(+3) + (-10) = -7$ and $(+3) \times (-10) = -30$

Then the expression can be seen to factorise as:
$x^2 - 7x - 30 = (x + 3)(x - 10)$

Remember you can check your answer by multiplying out the two brackets:

$(x + 3)(x - 10) = x^2 - 10x + 3x - 30 = x^2 - 7x - 30$

You also need to be able to factorise using the difference of two squares.

Consider expanding: $(x + 3)(x - 3)$
$= x^2 + 3x - 3x - 3^2$
$= x^2 - 3^2$
$= x^2 - 9$

As can be seen, this last line is a difference of two squares: x^2 and 3^2

Thus, to factorise the difference of these two squares, i.e. to factorise, $x^2 - 9$, we would write the above expressions out backwards:

$x^2 - 9$
$= x^2 - 3^2$
$= (x + 3)(x - 3)$

Example 2

Factorise $x^2 - 25$ using the difference of two squares.

$x^2 - 25$
$= x^2 - 5^2$
$= (x + 5)(x - 5)$

Example 3

Factorise
(a) $25x^2 - 81$ **(b)** $49y^2 - 121x^2z^4$

(a) $25x^2 - 81 = (5x)^2 - 9^2 = (5x + 9)(5x - 9)$
(b) $49y^2 - 121x^2z^2 = (7y)^2 - (11xz)^2 = (7y + 11xz)(7y - 11xz)$

Exercise 2A (Revision)

1. Factorise these linear expressions:
(a) $81 - 9t$ **(b)** $9x + 18y$ **(c)** $32xy - 8xz$
(d) $3p - 12q + 6$ **(e)** $144 - 12y + 4x$ **(f)** $5t - 10 + 15q$

2. Factorise these expressions using the difference of two squares:
(a) $x^2 - 49$ **(b)** $p^2 - q^2$ **(c)** $36 - 49y^2$
(d) $196 - 25x^2y^2$ **(e)** $81x^2t^2 - 169y^2$ **(f)** $16y^2 - 121$

3. Factorise the following quadratic expressions:
(a) $x^2 - 4x - 21$ **(b)** $x^2 + 5x + 6$ **(c)** $x^2 - x - 12$ **(d)** $x^2 - 7x + 12$ **(e)** $x^2 - 3x - 10$
(f) $x^2 + 18x + 80$ **(g)** $x^2 - 4x - 5$ **(h)** $x^2 + 7x - 18$ **(i)** $x^2 - 13x - 30$ **(j)** $x^2 - 8x - 20$
(k) $x^2 - 4x - 60$ **(l)** $x^2 - x - 110$

2.2 Factorising Differences of Two Squares With a Common Factor

When factorising a difference involving squares, sometimes there is a common factor in both terms that needs to be factorised out first before using the difference of two squares pattern.

Example 4

Factorise:
(a) $2x^2 - 8$ **(b)** $xy^2 - 25x$

(a) $2x^2 - 8 = 2(x^2 - 4) = 2(x^2 - 2^2) = 2(x + 2)(x - 2)$
(b) $xy^2 - 25x = x(y^2 - 25) = x(y^2 - 5^2) = x(y + 5)(y - 5)$

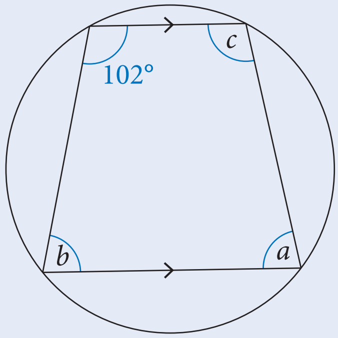
Sometimes more than one common factor may be taken out.

Example 5

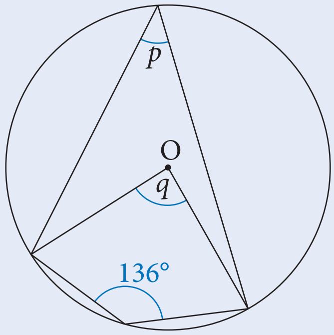
Factorise:
(a) $81wx^2 - 9w$ **(b)** $50ab^2 - 32a$

(a) $81wx^2 - 9w = 9w(3^2x^2 - 1) = 9w(3x + 1)(3x - 1)$
(b) $50ab^2 - 32a = 2a(25b^2 - 16) = 2a(5^2b^2 - 4^2) = 2a(5b + 4)(5b - 4)$

(c)



(d)



2. P, Q, R and S are points on a circle. RS is a diameter of the circle. Angle SPQ = 134°. Work out the size of angle QSR.

6.4 Radius and Tangent

A tangent to a circle meets the radius of the circle at 90°.

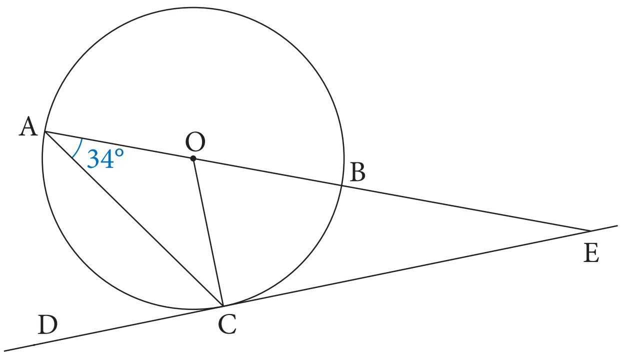
Example 6

The diagram on the right shows a circle, centre O. Line DE is a tangent to the circle, meeting the circle at point C. OC is a radius. Angle OAC = 34°.

(a) Explain why angles OCD and OCE are both 90°

(b) Explain why angle OCA = 34°

(c) Find the sizes of angles:
(i) ACD (ii) COB (iii) CEB



(a) Angles OCD and OCE are both 90° because they are both angles between the tangent DE and the radius OC.

(b) The two radii OA and OC are equal in length, so OAC is an isosceles triangle. Therefore, the angles OAC and OCA must be equal, and OCA = 34°.

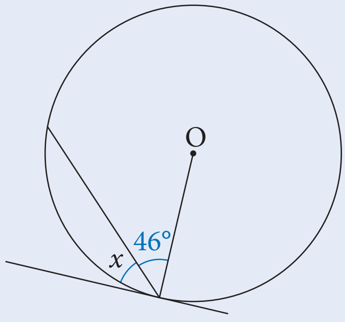
(c) (i) Angles ACD and OCA add up to 90°. So ACD = 90 – 34 = 56°.

(ii) Angle COB = 68°. It is double the angle OAC, since the angle at the centre is double the angle at the circumference. (Both angles are subtended by chord BC.)

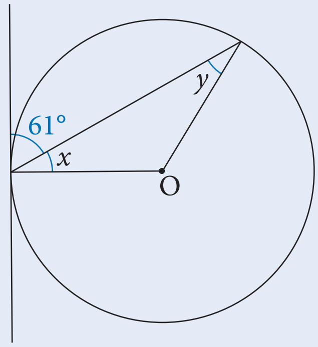
(iii) The three angles in triangle OCE must add up to 180°, so angle CEB = 180 – 90 – 68 = 22°.

Exercise 6D

1. In the diagram below, O is the centre of the circle. Find the size of x.



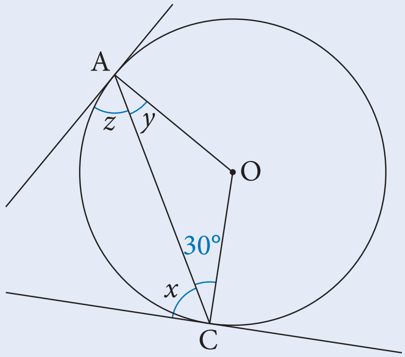
2. The diagram below shows a circle, centre O. Find the sizes of x and y.



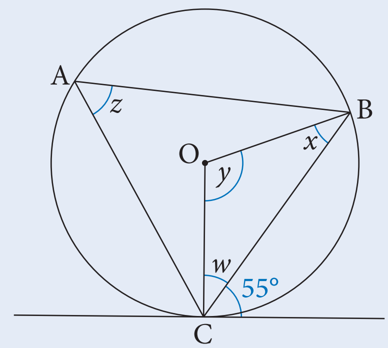
3. In the diagram below two tangents meet the circle with centre O.

(a) Explain why $y = 30°$.

(b) Find the sizes of the angles marked x and z.



4. The diagram below shows a circle with centre O. A tangent meets the circle at point C. Points A and B lie on the circumference of the circle. Find the sizes of the angles marked w, x, y and z.



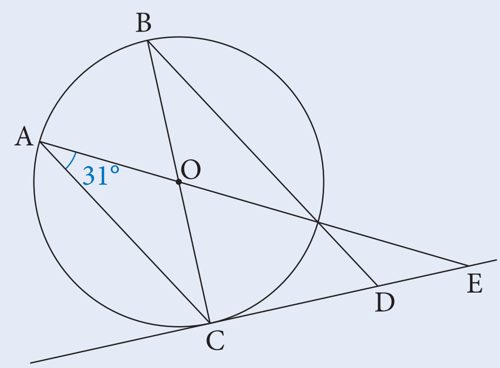
5. In the diagram shown on the right, O is the centre of the circle. A, B and C are points on the circumference of the circle. CDE is a tangent to the circle. Calculate the size of these angles:

(a) CBD

(b) COE

(c) OEC

(d) BDC

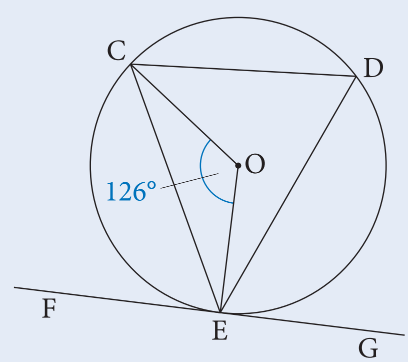


6. Points C, D and E lie on a circle, as shown in the diagram on the right. Line FG is a tangent to the circle, touching the circle at point E. Point O is the centre of the circle. Find the size of these angles:

(a) CDE

(b) CEO

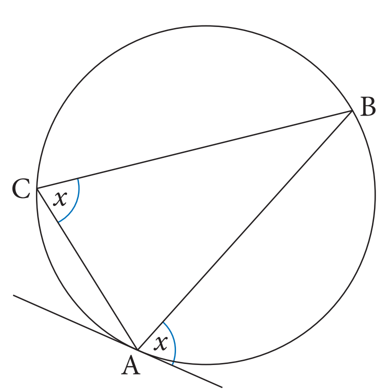
(c) CEF



6.5 Alternate Segment Theorem

The diagram shows a tangent touching a circle at point A. The chord AB divides the circle into two segments. The angle at point C on the circumference is subtended by chord AB. The two angles marked x are equal.

The alternate segment theorem states that the angle between the tangent and a chord is equal to the angle at the circumference in the opposite segment.



(b) The mass *m* grams of 28 small objects.

m (grams)	$0.5 < m \le 6.5$	$6.5 < m \le 9.5$	$9.5 < m \le 14.5$
Frequency	12	9	7

(c) The speeds of vehicles passing a checkpoint on the A1 one evening.

Speed *s* (mph)	$40 < s \le 50$	$50 < s \le 55$	$55 < s \le 60$	$60 < s \le 70$	$70 < s \le 90$
Frequency	12	15	11	8	4

(d) The number of people sitting at the tables in a restaurant one evening.

Number of people *x*	Frequency *f*
1 – 2	4
3 – 4	6
5 – 6	8
7 – 8	3

(e) The time taken for a group of 30 school pupils to complete a puzzle. The times are rounded to the nearest second.

Time *t* (s)	Frequency *f*
11 – 20	5
21 – 25	8
26 – 30	9
31 – 35	6
36 – 45	2

8.3 Interpreting Histograms

You may be given a histogram and asked to interpret it. Often this involves completing the frequency column in a grouped frequency table.

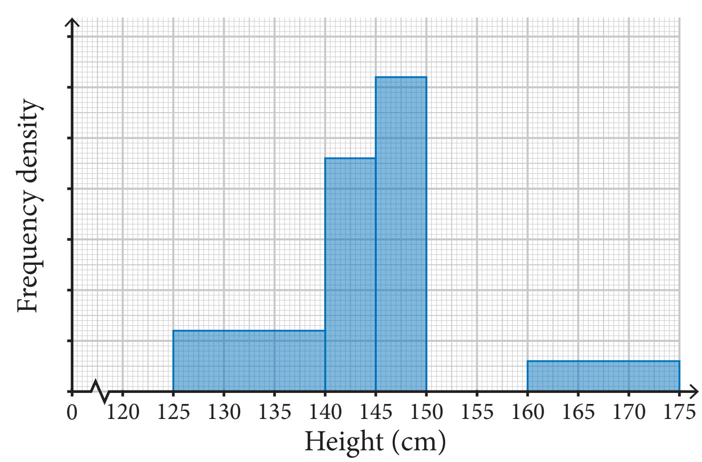
In the following example you are asked to draw only one of the bars. You must also complete the frequency table using the histogram.

Example 4

The histogram on the right shows the distribution of heights of 100 girls in year 10. One bar is missing.
Copy the histogram.

(a) There are 18 girls in the 125 – 140 cm group and 18 girls in the 150 – 160 cm group.

- **(i)** Complete the scale on the frequency density axis.
- **(ii)** Complete the histogram by adding the missing bar.



(b) Copy and complete the frequency table shown on the right.

Height *h* (cm)	Frequency
$125 \le h < 140$ cm	18
$140 \le h < 145$ cm	
$145 \le h < 150$ cm	
$150 \le h < 160$ cm	18
$160 \le h < 175$ cm	

(a) (i) There are 18 girls in the 125 – 140 cm group.
So the frequency is 18 and the class width is 15 cm.

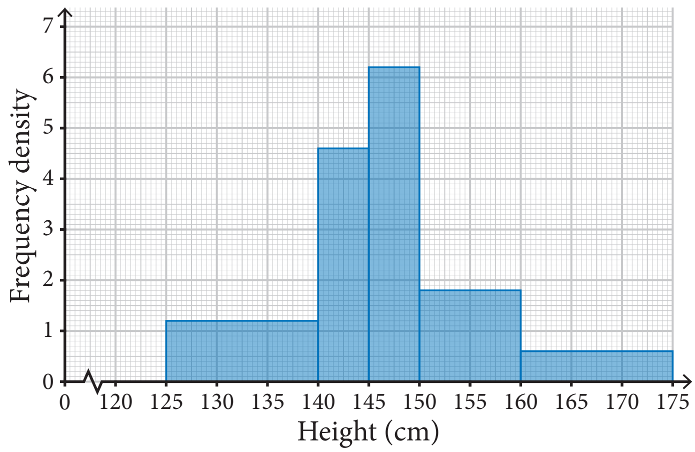
The frequency density is given by: Frequency density = $\frac{\text{Frequency}}{\text{Class width}} = \frac{18}{15} = 1.2$

Therefore, the first bar has a height of 1.2 units and the scale on the frequency density axis can be completed.

(ii) The missing bar relates to the 150 – 160 cm group.
The frequency is 18, so the area of the bar is 18
Its width is 10 cm.

The frequency density is given by: Frequency density = $\frac{\text{Frequency}}{\text{Class width}} = \frac{18}{10} = 1.8$

So the bar can be drawn with this height.



(b) **For the 140 – 145 cm group:**

Frequency = Class width × Frequency density
= 5 × 4.6
= 23

For the 145 – 150 cm group:

Frequency = Class width × Frequency density
= 5 × 6.2
= 31

For the 160 – 175 cm group:

Frequency = Class width × Frequency density
= 15 × 0.6
= 9

The frequency table can be completed as shown on the right.

Height h (cm)	Frequency
$125 \leq h < 140$cm	18
$140 \leq h < 145$cm	**23**
$145 \leq h < 150$cm	**31**
$150 \leq h < 160$cm	18
$160 \leq h < 175$cm	**9**

The following example demonstrates comparing two distributions.

Example 5

The histogram on the right shows the distribution of heights for the 1000 tallest buildings in central Toronto in Canada.

(a) Estimate the median height of these 1000 buildings.