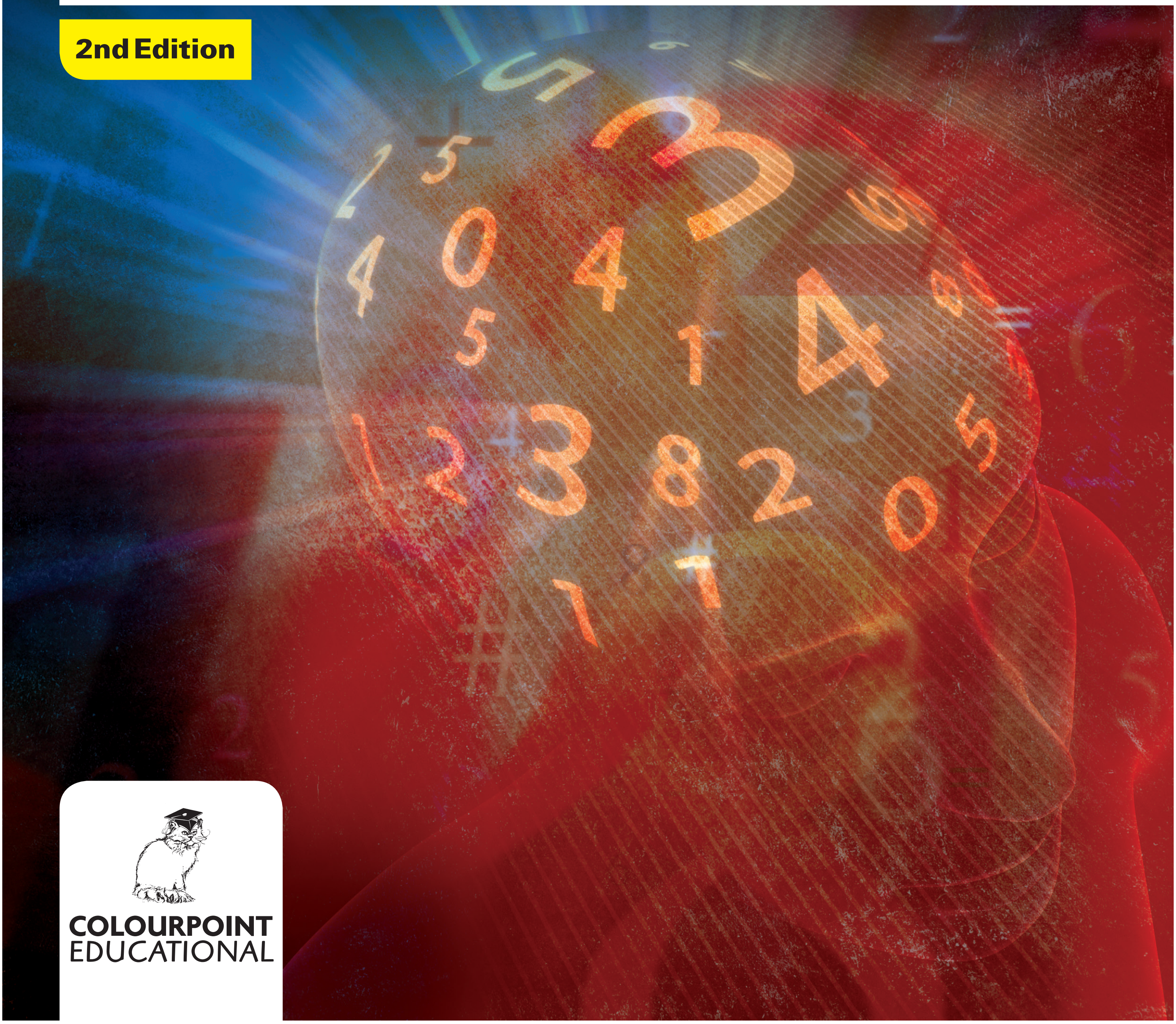
Luke Robinson

CCEA AS

PURE MATHEMATICS

2nd Edition



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Introduction

Why Study Mathematics?

Mathematics is all around us. It helps us to understand a complex and ever-changing world. It helps us to think analytically and, in turn with analytical thinking, we can investigate and know the truth about the world.

Many jobs require a high level of mathematical competence, such as engineering and various roles in the world of finance, statistics and computing. There are even data scientists who extract meaning from huge sets of data on population behaviour, such as voting patterns or internet usage. Even if you don't enter one of these fields of work, you will very likely use your AS and A2 Mathematics skills in your further studies and in your chosen career. Mathematics crops up in many surprising areas!

CCEA's AS and A2 Level Mathematics specifications are designed to prepare you for the mathematical content you will find in a wide range of different degree courses and occupations. Good luck!

Changes to the Specification

This book covers the revised specification for Unit AS 1: Pure Mathematics for CCEA, which was available for teaching from September 2018 onwards.

The following changes from the previous specification are worth paying particular attention to.

Problem solving

The new specification has a greater emphasis on **problem solving** and the final chapter of this book (Chapter 14: Problem Solving) has been included to address this. This chapter includes material explaining what a problem solving task will look like, as well as examples and an exercise of practice questions.

Problem solving questions may require techniques from any of the preceding chapters, and often more than one. They may also require understanding of the mathematics taught at GCSE level.

Straight line graphs in context

The new specification includes questions on straight line graphs set in context. In all questions set in a context, answers to problems should be expressed in terms of the context of the question. For more details see section 6.5 of Chapter 5: Straight Lines.

Proofs

Fewer proofs are required at AS Level in the new specification. The complete list of proofs required for AS Pure Mathematics is as follows.

- Addition law for logarithms:
  $\log_a x + \log_a y = \log_a (xy)$
- Subtraction law for logarithms:
  $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$
- Power law for logarithms:
  $n \log_a x = \log_a x^n$

These three proofs are detailed in Chapter 10: Exponentials and Logarithms.

Calculators

Calculators with new features, such as the Casio FX-991EX, are now available. These new features include:

- A quadratic and cubic equation solver.
- Evaluation of definite integrals.
- Evaluation of summations.

All of the functions on these calculators can be used in the AS and A2 examinations, but pay attention to the exact wording of the question. For example, if a question asks you to "show all the steps in your working", or states that "solutions relying on calculator technology will not be accepted", or "using algebra", then you will lose marks if you rely entirely on the calculator for a calculation. In these circumstances, you can still use any of these new features on your calculator to check your answer.

Calculator tips are given throughout this book.

Chapter 1
Indices and Surds

1.1 Introduction

Key words

- **Index (plural indices)**: A power, for example the power 3 in 5^3.
- **Index form**: A number or algebraic expression written using an index, for example 5^3 or x^2.
- **Surd**: A root (for example a square root or a cube root) of a number is called a surd if its value is an irrational number.

Before you start

You should know:

- How to recognise some common squares and cubes, e.g.
  $4^2 = 4 \times 4 = 16$
  $2^3 = 2 \times 2 \times 2 = 8$
- How to evaluate powers of 2, e.g.
  $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$
- How to use the difference of two squares:
  $a^2 - b^2 = (a - b)(a + b)$
- How to recognise rational and irrational numbers. Rational numbers are either integers, numbers with terminating decimals or recurring decimals. A rational number can be written as a fraction, with integers in both the numerator and denominator; irrational numbers cannot. Some examples are:
  Rational: ½, $1.\dot{6}$, 3, -8, 2.888
  Irrational: π, 5π, $\sqrt{2}$, $\sqrt[3]{3}$, $\dfrac{5}{\sqrt{10}}$

Exercise 1A (Revision)

1. Evaluate:
   (a) The squares of 4, 12, 15 and 20
   (b) 3^3, 5^3, 10^3
   (c) $\sqrt{81}$, $\sqrt{6^2 + 8^2}$, $\sqrt{1.44}$, $\sqrt{169}$, $(\sqrt{17.2})^2$
   (d) The cube roots of 64 and 8
   (e) 2^n where $n = 2, 3, 6, 9$

Exercise 1A...

2. Use the difference of two squares to rewrite the following:
   (a) $x^2 - 9$ (b) $a^2 - b^2$
   (c) $1 - c^2$ (d) $(d + 10)(d - 10)$
   (e) $e^2f^2 - (gh)^2$
3. Are these numbers rational or irrational?
   (a) 1.15 (b) $1 + \pi$
   (c) $0.\dot{2}\dot{6}$ (d) $\sqrt{3} - 10$

What you will learn

In this chapter you will learn:

- How to understand and use **index notation**.
- More about using the **laws of indices**.

This chapter provides the basis for a lot of the pure mathematics used in AS and A2 Mathematics. You will use indices a lot in the chapters on differentiation and integration. You will learn how to simplify expressions and solve equations using the rules of indices (powers). Some of this section will be revision of the work you did at GCSE.

1.2 Index Notation

5^3 ('5 cubed') and 7^2 ('7 squared') are examples of numbers in index form.

The power 2 in 7^2 and the power 3 in 5^3 are known as **indices**. Indices are useful – for example, they allow us to represent numbers in standard form – and have a number of important properties. As a reminder:

$7^1 = 7$
$7^2 = 7 \times 7$
$7^3 = 7 \times 7 \times 7$ and so on.

Note: If you are asked to evaluate an expression, you should give your final answer as a number, e.g. 32 or $\frac{1}{5}$.
If you are asked to give an answer in index form, you will give an answer such as 2^5 or x^{-7}.

Exercise 1B

1. Evaluate:
   (a) 2^5 (b) 3^4 (c) 4^3
   (d) 14^2 (e) $2^3 + 3^2$ (f) $3^4 + 4^3$
   (g) $6^2 + 5^2$ (h) $(3^2)^2$ (i) $\left(\frac{2}{3}\right)^3$
   (j) $(0.2)^2$
2. Write the following in index form. There may be more than one way.
   (a) 49 (b) 100 (c) 0.01
   (d) 121 (e) 0.001 (f) $\frac{9}{49}$
   (g) 0.09 (h) $\frac{1}{100000}$ (i) 0.16

In the real world...

The population of the world is roughly 7.6 billion or 7 600 000 000 people. Doesn't it look much easier and neater to write 7.6×10^9?

How much storage space is on your USB pen drive? 16 GB is 1.6×10^{10} bytes.

1.3 The Laws of Indices

There are several rules for dividing and multiplying numbers written in index form. These properties only hold, however, when the same base is being used. For example, we cannot easily work out what $2^3 \times 5^2$ would be, but we can simplify $3^2 \times 3^3$.

Multiplication

When we multiply numbers with indices, we add the powers. So for example:

$$z^a \times z^b = z^{a+b}$$

Remember, this doesn't work if the base changes. For example, we cannot simplify:

$$z^a \times w^b$$

There is no easy way of simplifying:

$$7^5 \times 2^{-3}$$

because 7 and 2 are different bases. This is true for all our laws of indices.

Worked Example

1. Simplify:
   (a) $x^2 \times x^6$ (b) $5^5 \times 5^{-2}$ (c) $2y^2 \times 4y^5$

(a) $x^2 \times x^6 = x^8$
(b) $5^5 \times 5^{-2} = 5^3$ (because 5 + (−2) = 3)
(c) $2y^2 \times 4y^5$
$= 2 \times 4 \times y^2 \times y^5$
$= 8y^7$ (adding the powers of y)

Division

If we divide two numbers with indices, we subtract the powers. So for example:

$$p^a \div p^b = p^{a-b}$$

Worked Example

2. Simplify:
   (a) $\frac{r^2}{r^3}$ (b) $t^2 \div t^3$
   (c) $6^2 \div 6^{-5}$ (d) $6z^6 \div 2z^2$

(a) $\frac{r^2}{r^3} = r^{-1}$
(b) $t^2 \div t^3 = t^{-1}$
(c) $6^2 \div 6^{-5} = 6^7$
(d) $6z^6 \div 2z^2$
$= (6 \div 2) \times (z^6 \div z^2)$
$= 3z^4$

Brackets

If we have a number with an index, all raised to another power, this is the only time we multiply our indices:

$$(x^a)^b = x^{ab}$$

Worked Example

3. Simplify:
   (a) $(x^2)^3$ (b) $(5^3)^2$
   (c) $9^3 \times 3^4$ (d) $(3p^2)^3 \div p^4$

(a) $(x^2)^3 = x^6$
(b) $(5^3)^2 = 5^6$
(c) As noted above, you cannot simplify an expression involving indices if the bases are different. However, sometimes it is possible to make the bases the same:
$9^3 \times 3^4 = (3^2)^3 \times 3^4$
$= 3^6 \times 3^4$
$= 3^{10}$
(d) $(3p^2)^3 \div p^4$
$= 27p^6 \div p^4$
$= 27p^2$

Exercise 1C

1. Simplify these expressions:
   (a) $x^2 \times x^4$ (b) $2x^2 \times 4x^{-3}$
   (c) $6p^3 \div 2p^2$ (d) $6r^3 \times 2r^{-2}$
   (e) $8s^3 \div 2s^{-2}$ (f) $(3a^3)^3 \div a^4$
   (g) $(3b^3)^2 \div b^2$ (h) $6c^3 \times 2c^2 \times c$
   (i) $6r^3 \times 2r^2 \div r^2$ (j) $(4d^3)^3 \div 2d^4$

Negative indices

A negative index denotes a **reciprocal**.

Worked Example

4. Simplify:

(a) n^{-1} (b) n^{-a} (c) 3^{-2} (d) $\left(\frac{1}{2}\right)^{-3}$

(a) $n^{-1} = \frac{1}{n}$

(b) $n^{-a} = \frac{1}{n^a}$

(c) $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

(d) $\left(\frac{1}{2}\right)^{-3} = 2^3 = 8$

(Remember: the reciprocal of $\frac{1}{2}$ is 2)

Note: A negative power does not make your answer negative!

$2^{-1} = \frac{1}{2}$, not $-\frac{1}{2}$

The power of zero

Anything to the power 0 is equal to 1. The table below may help you see why this is true. Looking at the sequence of numbers in the second row, try to fill in the missing number.

3^{-1}	3^0	3^1	3^2	3^3
$\frac{1}{3}$		3	9	27

Worked Example

5. Simplify:

(a) 4^0 (b) $(-124)^0$ (c) x^0

(a) $4^0 = 1$
(b) $(-124)^0 = 1$
(c) $x^0 = 1$

Fractional indices

If the index is a fraction, we must take a root of the number. For example, $4^{1/2}$ means take the square root of 4. Similarly, an index of ⅓ means take the cube root.

Usually we consider the square root sign √ and the power of ½ both to mean the positive square root.

Worked Example

6. Evaluate:

(a) $\left(\frac{9}{64}\right)^{1/2}$ (b) $8^{1/3}$
(c) $8^{-1/3}$ (d) $\left(\frac{1}{8}\right)^{-1/3}$

(a) $\left(\frac{9}{64}\right)^{1/2} = \sqrt{\frac{9}{64}} = \frac{\sqrt{9}}{\sqrt{64}} = \frac{3}{8}$

(b) $8^{1/3} = \sqrt[3]{8} = 2$ (the cube root of 8)

(c) $8^{-1/3} = \left(\frac{1}{8}\right)^{1/3} = \frac{1}{2}$ (the cube root of $\frac{1}{8}$)

(d) $\left(\frac{1}{8}\right)^{-1/3} = 8^{1/3}$ (take the reciprocal of the fraction while making the index positive) = 2

More complicated fractional indices

Sometimes the index is a more complicated fraction. For example, if it is ⅔ we must take the cube root of the number, then raise it to the power 2. For example, $8^{2/3}$ means take the cube root of 8, which is 2, then square it, which gives 4.

Note: The operations can be reversed: you can square first and then perform the cube root. However, performing the cube root first is often easier when working without a calculator.

In general:

$$a^{p/q} = \left(\sqrt[q]{a}\right)^p = \sqrt[q]{a^p}$$

You may be asked to write an index expression using surd notation. We will study surds in more detail in the next section.

Worked Examples

7. Rewrite using surd notation:

(a) $a^{2/3}$ **(b)** $a^{5/2}$ **(c)** $2^{2/3}$

(a) $a^{2/3} = \left(\sqrt[3]{a}\right)^2$

(b) $a^{5/2} = \left(\sqrt{a}\right)^5$

(c) $2^{2/3} = \left(\sqrt[3]{2}\right)^2$

8. Evaluate $8^{2/3}$

Using the result from example 7(a):

$8^{2/3} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$

9. Simplify $\dfrac{4x^{3/2}}{8x}$

Cancel down the fraction 4⁄8 and subtract the indices.

$\dfrac{4x^{3/2}}{8x} = \dfrac{1}{2}x^{1/2}$

Exercise 1D

1. Evaluate the following:

(a) $25^{1/2}$ **(b)** $8^{4/3}$ **(c)** $16^{-3/2}$

(d) $16^{5/4}$ **(e)** $27^{-4/3}$ **(f)** 5^{-2}

(g) $4^{-2} \times 4^{-3}$ **(h)** 64^0 **(i)** $\left(\dfrac{1}{16}\right)^{1/2}$

(j) $\left(\dfrac{1}{64}\right)^{-1/2}$ **(k)** $64^{2/3}$ **(l)** $\left(1\dfrac{9}{16}\right)^{3/2}$

(m) $(-4)^{-3}$

2. Simplify these expressions:

(a) $g^{-3} \times g^{-3}$ **(b)** $\dfrac{1}{t^{-2}}$

(c) $(q^3)^0$ **(d)** $(f^{3/2})^2$

(e) $(b^{1/a})^a$ **(f)** $\dfrac{4x^{6/5}}{20x}$

(g) $24x^{1\frac{1}{4}} \div 3x^{1/4}$ **(h)** $10x^{5/2} \div 5x$

(i) $\dfrac{18x^{4/3}}{3x} \div \dfrac{45x^{4/3}}{5x}$ **(j)** $10x^{5/2} \times (4x^{3/4})^2$

(k) $30x^{5/4} \times 3x$ **(l)** $(4x)^{5/2} \div 4x$

(m) $-7x^{3/2} \div 14x^{5/2}$ **(n)** $(27x^2)^{2/3} \div 3x$

Solving equations

You may be asked to solve equations in which the unknown is in one or more of the indices.

Attempt to make the bases the same on both sides of the equation. Then you will use the fact that:

If $a^b = a^c$

then $b = c$

This is called equating the indices.

Worked Examples

10. Solve for y: $8^y = 64^4$

$8^y = (8^2)^4$
$8^y = 8^8$
When the bases are the same, equate the indices:
$y = 8$

11. Solve for x: $\dfrac{6^x}{36^{x-2}} = \sqrt{6}$

$6^x = (\sqrt{6})(36^{x-2})$

$6^x = (6^{1/2})(6^2)^{x-2}$

$6^x = (6^{1/2})6^{2x-4}$

$6^x = 6^{2x-7/2}$

Equating indices:

$x = 2x - 7/2$

$x = 7/2$

Exercise 1E

1. Solve for the variable in the equation:

(a) $\sqrt{2} = \dfrac{2^g}{4^2}$ **(b)** $\sqrt{3} \times 3^t = 9^4$

(c) $\dfrac{\sqrt{3}}{3^{5f+2}} = 3^{-1/4}$ **(d)** $2^y = 4^5 \times \sqrt{2}$

(e) $3^k = \dfrac{3^{2k+6}}{\sqrt{3}}$ **(f)** $\dfrac{\sqrt{2}}{4^2} = 2^d$

(g) $\left(\sqrt{2}\right)^w = \dfrac{2^w}{4^3}$ **(h)** $\sqrt{3} = \dfrac{3^{4q+3}}{9^5}$

(i) $\dfrac{4^2}{2^z} = \sqrt{2}$ **(j)** $4^3 \times 2^{6g+6} = 2^g$

1.4 Surds

Introduction

A root (for example a square root or a cube root) of a number is called a **surd** if its value is an irrational number.

Some examples of surds are $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt[3]{2}$, and $\sqrt[3]{3}$.

However, there are many roots that are **not surds**:

$\sqrt[3]{27}$ (this is equal to 3, which is a rational number)

$\sqrt{4}$ (this equals 2)

$\sqrt{\frac{9}{4}}$ (this equals $\frac{3}{2}$, which is rational)

Surds are often left in the solutions to equations, for example:

$x = 1 + \sqrt{3}$ or $z = \frac{\sqrt[3]{2}}{2}$

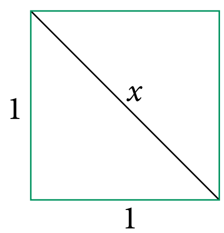
In this way we can give more accurate answers to some problems. You will often use surds when you need to give an exact answer to a problem. For example, they are often used when solving quadratic equations.

Worked Example

12. What is the length of the diagonal of a square whose sides are 1 cm? Give your answer in surd form.

Using Pythagoras' Theorem:

$x = \sqrt{1^2 + 1^2} = \sqrt{2}$ cm



In the real world...

One very famous use of surds is in the Golden Ratio. The Golden Ratio appears frequently in art and nature, as well as in mathematics and science. It is sometimes given the Greek letter Φ and has the equation:

$$\Phi = \frac{1 + \sqrt{5}}{2} \approx 1.61$$

When it is used in art and architecture, the Golden Ratio is said to give the most beautiful results. The Acropolis in Greece was built over 2000 years ago. Some studies of this ancient building suggest that the Golden Ratio was used throughout, for example the width divided by the height of the front façade.

We also see the Golden Ratio in flower petals, pine cones, cauliflower florets, snails' shells, weather systems, etc.

Among mathematicians of the ancient world, the square root of 5 was controversial. It is an irrational number, so whenever you give its value to say 2 decimal places or 3 significant figures, you have introduced an error. Many mathematicians of the ancient world refused to believe irrational numbers even existed. We now know it is often best to leave an answer in surd form, e.g. $1 + \sqrt{5}$. This way it is exact, without any rounding error.

Use and manipulation of surds

In this section you will learn how to use and manipulate surds. There are some important rules when manipulating surds:

Rule 1: $\sqrt{a \times b} = \sqrt{a}\sqrt{b}$

Rule 2: $\sqrt{a} \times \sqrt{a} = a$

Rule 3: $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Rule 4: $a\sqrt{b} \pm c\sqrt{b} = (a \pm c)\sqrt{b}$

Worked Example

13. Simplify the following expressions involving surds.

(a) $\sqrt{98}$ **(b)** $\sqrt{\frac{121}{100}}$ **(c)** $\sqrt{48} + \sqrt{108}$

(d) $\sqrt{30}$ **(e)** $\sqrt{5} \times \sqrt{5}$

(a) When simplifying a surd $\sqrt{a}$, find the biggest square number that is a factor of a. Then use Rule 1. In this case 49 is the biggest square number that is a factor of 98. So:

$$\begin{aligned}\sqrt{98} &= \sqrt{49}\sqrt{2}\\ &= 7\sqrt{2}\end{aligned}$$

(b) Using Rule 3:

$$\begin{aligned}\sqrt{\frac{121}{100}} &= \frac{\sqrt{121}}{\sqrt{100}}\\ &= \frac{11}{10}\end{aligned}$$

(c) Using Rule 1 for both surds:

$$\begin{aligned}\sqrt{48} + \sqrt{108} &= \sqrt{16}\sqrt{3} + \sqrt{36}\sqrt{3}\\ &= 4\sqrt{3} + 6\sqrt{3}\\ &= 10\sqrt{3} \text{ (using Rule 4)}\end{aligned}$$

(d) $\sqrt{30}$ – we leave this unchanged because there are no square numbers that are factors of 30.

(e) Using Rule 2:

$\sqrt{5} \times \sqrt{5} = 5$

At the beginning of this chapter you revised the difference of two squares:

$$a^2 - b^2 = (a - b)(a + b)$$

This can be applied to some surd problems.

Worked Example

14. Simplify: $(7 - \sqrt{5})(7 + \sqrt{5})$

This is an expression of the form $(a - b)(a + b)$.

We can use the difference of two squares:

$$(a - b)(a + b) = a^2 - b^2$$

$$\begin{aligned}(7 - \sqrt{5})(7 + \sqrt{5}) &= 7^2 - (\sqrt{5})^2 \\ &= 49 - 5 \\ &= 44\end{aligned}$$

$7 + \sqrt{5}$ and $7 - \sqrt{5}$ are known as conjugate surds.

Rationalising the denominator

Usually, we do not leave a surd in the denominator of a fraction. For example, we would **not** give the answer to a question like this:

$$y = \frac{1}{\sqrt{3}}$$

or this:

$$x = \frac{4}{\sqrt{5} + 2}$$

Instead, we must multiply both numerator and denominator by something to make the surd appear only in the numerator.

Method 1: If the denominator is a simple surd, multiply top and bottom by this surd.

Method 2: If the denominator is of the form $a \pm \sqrt{b}$, multiply the top and bottom by the denominator's conjugate surd.

Worked Examples

15. Rationalise the denominator and simplify: $\dfrac{6}{\sqrt{3}}$

$$\begin{aligned}\frac{6}{\sqrt{3}} &= \frac{6 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} \\ &= \frac{6\sqrt{3}}{3} \\ &= 2\sqrt{3}\end{aligned}$$

16. Simplify: $\dfrac{4 - \sqrt{5}}{\sqrt{5} - 2}$

giving your answer in the form $a\sqrt{5} + b$, where a and b are integers.

$$\begin{aligned}\frac{4 - \sqrt{5}}{\sqrt{5} - 2} &= \frac{(4 - \sqrt{5})(\sqrt{5} + 2)}{(\sqrt{5} - 2)(\sqrt{5} + 2)} \\ &= \frac{4\sqrt{5} - 5 + 8 - 2\sqrt{5}}{5 - 2\sqrt{5} + 2\sqrt{5} - 4} \\ &= \frac{2\sqrt{5} + 3}{1} \\ &= 2\sqrt{5} + 3\end{aligned}$$

Exercise 1F

1. Simplify:

(a) $\sqrt{32}$ **(b)** $\dfrac{\sqrt{27}}{3}$

(c) $\sqrt{44} \div \sqrt{11}$ **(d)** $\sqrt{1\frac{21}{100}}$

2. Express $\sqrt{450}$ in the form $a\sqrt{2}$ where a is an integer.

3. Express $\sqrt{180}$ in the form $a\sqrt{5}$ where a is an integer.

4. Express $(5 - \sqrt{2})^2$ in the form $b + c\sqrt{2}$ where b and c are integers.

5. Express $(6 - \sqrt{5})^2$ in the form $b + c\sqrt{5}$ where b and c are integers.

6. Express $\sqrt{18}$ in the form $a\sqrt{2}$ where a is an integer.

7. Express $\sqrt{245}$ in the form $a\sqrt{5}$ where a is an integer.

8. Express $\dfrac{3(2 + \sqrt{2})}{2 - \sqrt{2}}$ in the form $b + c\sqrt{2}$ where b and c are integers.

Exercise 1F...

9. Express $\dfrac{2(3+\sqrt{5})}{3-\sqrt{5}}$ in the form $b + c\sqrt{5}$ where b and c are integers.

10. Rationalise the denominator and simplify:

(a) $\dfrac{1}{\sqrt{5}}$ (b) $\dfrac{\sqrt{3}}{\sqrt{15}}$ (c) $\dfrac{1+\sqrt{5}}{\sqrt{7}}$

11. Giving your answers in the form $a + b\sqrt{2}$, where a and b are rational numbers, find:

(a) $(6-\sqrt{8})^2$ (b) $(1+\sqrt{8})^2$

(c) $\dfrac{1}{5-\sqrt{8}}$ (d) $\dfrac{1}{3+\sqrt{8}}$

12. Giving your answers in the form $a + b\sqrt{3}$ where a and b are rational numbers, find:

(a) $(7-\sqrt{27})^2$ (b) $\dfrac{1}{6-\sqrt{27}}$

13. Expand and simplify:

(a) $(4+\sqrt{2})(4-\sqrt{2})$

(b) $(7+\sqrt{2})(7-\sqrt{2})$

(c) $(4+\sqrt{3})(4-\sqrt{3})$

(d) $(6+\sqrt{3})(6-\sqrt{3})$

14. Express in the form $a + b\sqrt{c}$ where a, b and c are integers.

(a) $\dfrac{6}{3+\sqrt{6}}$ (b) $\dfrac{20}{3+\sqrt{5}}$

(c) $\dfrac{24}{4+\sqrt{8}}$ (d) $\dfrac{28}{2+\sqrt{2}}$

15. Simplify:

(a) $\sqrt{20}+\sqrt{80}$ (b) $\sqrt{12}+3\sqrt{48}$

(c) $\dfrac{\sqrt{125}+\sqrt{45}}{\sqrt{125}-\sqrt{45}}$ (d) $\dfrac{\sqrt{3}-\sqrt{7}}{\sqrt{3}+\sqrt{7}}$

1.5 Summary

The laws of indices are:

Rule 1 $a^p \times a^q = a^{p+q}$

Rule 2 $a^p \div a^q = a^{p-q}$

Rule 3 $(a^p)^q = a^{pq}$

Rule 4 $a^0 = 1$

Rule 5 $a^{-p} = \dfrac{1}{a^p}$

Rule 6 $a^{\frac{1}{p}} = \sqrt[p]{a}$

Rule 7 $a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$

You can use the following rules to manipulate surds:

Rule 1 $\sqrt{a \times b} = \sqrt{a}\sqrt{b}$

Rule 2 $\sqrt{a} \times \sqrt{a} = a$

Rule 3 $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

Rule 4 $a\sqrt{b} \pm c\sqrt{b} = (a \pm c)\sqrt{b}$

If there is a surd in the denominator of a fraction, you can **rationalise the denominator** by multiplying top and bottom by a surd expression:

Method 1: If the denominator is a simple surd, multiply top and bottom by this surd.

Method 2: If the denominator is the sum of a surd and a number, $a + \sqrt{b}$, multiply top and bottom by $a - \sqrt{b}$.