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CCEA A2

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Contents

Introduction

This book covers the revised specification for Unit A2 2: Applied Mathematics (Mechanics and Statistics) for CCEA, which was available for teaching from September 2018 onwards.

Accuracy

It is important to remember that all answers should be given either exact, or rounded to 3 significant figures. This advice is printed on the front page of all A Level Mathematics papers. Answer marks can be lost for rounding to any other level of accuracy.

Modelling

An important part of Applied Mathematics is **modelling**. Modelling questions may be set in relation to all topics in A2 Applied Mathematics.

What does a modelling question look like?

A modelling question typically involves several of the following features, but not necessarily all of them:

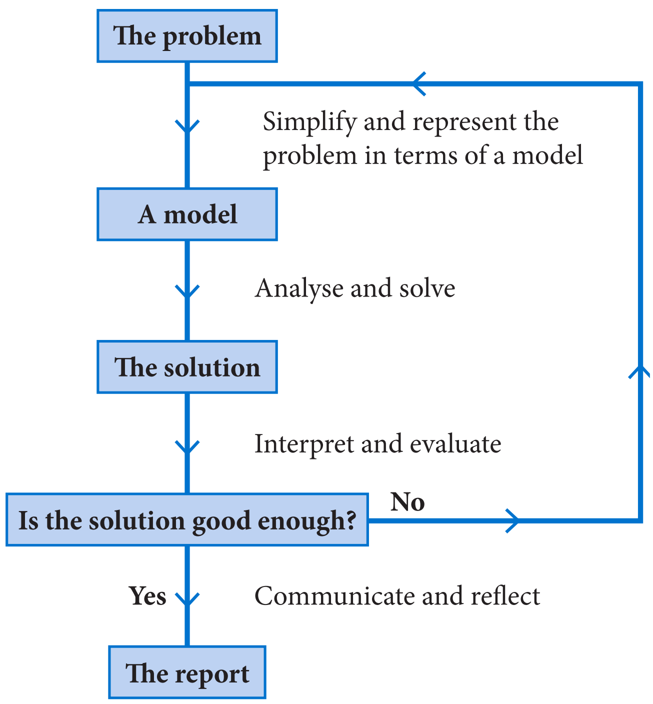
- There may be a requirement to make simplifications. The question will ask what simplifications or assumptions have been made.
- The candidate may be required to discuss the limitations of the model used.
- There may also be a requirement to refine or adapt the model or to consider different models.

The Modelling Cycle

The **Modelling Cycle** is outlined in the diagram. From the wording of the problem, the student should devise a way to model the situation. Simplifications and assumptions may be required.

The model should be applied to obtain a solution and this solution is interpreted and evaluated. At this point, it may become clear that certain assumptions were inappropriate, wrong, or not needed. It may be the case that different assumptions are required. In this way the model can be refined, and this modified version of the model is applied to the problem.

The final report should detail results, conclusions, any assumptions made and any limitations of the model being used. In A2 Level Mathematics, the report will comprise the solution to the problem.



The Modelling Cycle

Chapter 1
Kinematics

1.1 Introduction

Kinematics is the study of the motion of bodies without consideration of the forces that cause them to move. Therefore, in this chapter, there will be no calculations involving forces.

In AS Mathematics you studied kinematics with constant acceleration. In A2 Mathematics, motion with variable acceleration is considered.

Key words

- **Displacement**: The change in position of an object, expressed as a vector.
- **Distance**: The magnitude of the displacement vector.
- **Velocity**: The speed of an object in a given direction. Velocity is a vector quantity.
- **Speed**: The magnitude of the velocity vector.
- **Acceleration**: The rate of change of speed, or the rate of change of velocity.

Before you start

You should:

- Be familiar with the constant acceleration formulae (suvat formulae) and how to use them.
- Be able to apply the differentiation and integration techniques you learnt in AS and A2 Pure Mathematics.
- Understand vectors in two dimensions.

Exercise 1A (Revision)

1. A postman runs away from a dog down a garden path that is 10 metres long. He accelerates at 1 m s^{-2} for 3 seconds. Assuming he starts from rest, determine whether the postman reaches the gate at the end of the path in this time. Show all your working.

2. $y = 8t^2 + 14 + \dfrac{2}{t}$ where $0 < t \leq 4$

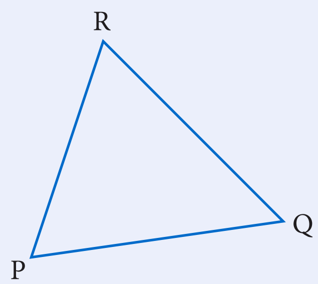
(a) Find $\dfrac{dy}{dt}$ in terms of t.

(b) Find the value of t that gives a minimum value of y.

(c) Find this value of y.

(d) Show that this is a minimum value of y, not a maximum.

3. The diagram shows a triangle PQR.



$\overrightarrow{PQ} = 7\mathbf{i} + \mathbf{j}$, $\overrightarrow{QR} = -5\mathbf{i} + 5\mathbf{j}$ and $\overrightarrow{RP} = -2\mathbf{i} - 6\mathbf{j}$

(a) Show that the triangle PQR is isosceles.

(b) Find the area of the triangle.

Notation

- A dot above the name of a variable denotes its first derivative with respect to time. For example, $\dot{\mathbf{r}}$ means $\dfrac{d\mathbf{r}}{dt}$
- Since, in this chapter, the displacement, velocity and acceleration may be functions of time, you may see the notation $s(t)$, $v(t)$ and $a(t)$.

What you will learn

In this chapter you will learn about:

- Motion in a straight line with variable acceleration.
- Motion in 2 dimensions with variable acceleration.

In the real world...

When a rocket is launched, a large part of its mass is the fuel that it carries. As soon as the rocket engines are ignited, fuel is burnt and the overall mass begins to fall.

To work out the rocket's acceleration, Newton's second law $F = ma$ could be rearranged to $a = \frac{F}{m}$. This tells us that the acceleration depends on both the mass and the upwards thrust being provided by the engines.

Since the mass is constantly changing as the fuel is burnt, the acceleration is constantly changing as well.

This is an example of **variable acceleration**.

Throughout this chapter you will encounter different systems in which the acceleration is not constant, but is dependent on the time since the object started moving. Typically, to find the distance travelled and the velocity at any time, integration and differentiation are required.

1.2 Motion in a Straight Line With Variable Acceleration

When a body experiences **variable acceleration**, you can model the acceleration as a function of time. You can use calculus to describe the relationship between displacement, velocity and acceleration.

You may have to use any of the functions and techniques from your A-Level pure maths course to analyse motion in a straight line with variable acceleration.

The variables involved in this section are:

s the displacement

v the velocity

a the acceleration

t time

When the acceleration is variable, you cannot use the constant acceleration formulae you learnt in AS Mathematics.

Instead use calculus, as summarised in the following diagram:

$$\begin{array}{c} \text{Differentiate} \\ \boldsymbol{s} \;\overset{\longrightarrow}{\longleftarrow}\; \boldsymbol{v} \;\overset{\longrightarrow}{\longleftarrow}\; \boldsymbol{a} \\ \text{Integrate} \end{array}$$

To put this information into equation form:

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} \quad \text{and} \quad a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$$

$$s = \int v \,\mathrm{d}t \quad \text{and} \quad v = \int a \,\mathrm{d}t$$

For example, to obtain an expression for the velocity, it is possible to differentiate an expression for the displacement with respect to time.

If a question involves integration, remember to include a constant of integration, as shown in the next example. In many cases the question will provide information that will allow you to calculate this constant.

Worked Example

1. A particle moves such that its velocity v m s^{-1} depends upon time t seconds according to $v = t^2 - 3t$

The particle is initially at rest at a fixed point O.

(a) Find an expression for the particle's displacement from O in terms of t.

(b) Find the particle's displacement from O when $t = 1$.

(c) Find an expression for the particle's acceleration in terms of t.

(d) Find the time at which the acceleration is zero.

(a)
$$s = \int v \,\mathrm{d}t$$
$$= \int t^2 - 3t \,\mathrm{d}t$$
$$s = \frac{1}{3}t^3 - \frac{3}{2}t^2 + c$$

The particle is initially at point O. In other words when $t = 0$, $s = 0 \therefore c = 0$.

$$\therefore s = \frac{1}{3}t^3 - \frac{3}{2}t^2$$

(b) When $t = 1$:

$$s = \frac{1}{3}(1)^3 - \frac{3}{2}(1)^2$$
$$s = -\frac{7}{6} \text{ m, or } -1.17 \text{ m (3 s.f.)}$$

Note: Remember displacement can be negative. If a question asks for distance, this is always a positive value. In this case the distance from O would be 1.17 m.

(c)
$$v = t^2 - 3t$$
$$a = \frac{\mathrm{d}v}{\mathrm{d}t}$$

$\therefore a = 2t - 3$

(d) To find the time at which $a = 0$:

$2t - 3 = 0$

$t = 1.5\,\text{s}$

The following two examples demonstrate how to find the total distance travelled by a particle in a certain time interval. Care must be taken if the particle's velocity is zero at some point within this time interval.

Worked Examples

2. A particle P is moving so that its velocity v m s^{-1} after t seconds is given by: $v = 3t^2 - 6t$

Initially P is at rest and has a displacement of 6 m from a fixed point O.

(a) Find v when $t = 1$

(b) Find an expression for the displacement of P from O at any time t.

(c) Find the total distance travelled by the particle by the time it returns to its initial position.

(a) When $t = 1$, $v = 3(1)^2 - 6(1) = -3\,\text{m s}^{-1}$

(b) $s = \int v\,\text{d}t$

$s = \int 3t^2 - 6t\,\text{d}t$

$s = t^3 - 3t^2 + c$

Initially (when $t = 0$), $s = 6$

$\therefore 6 = (0)^3 - 3(0)^2 + c$

$c = 6$

Therefore:

$s = t^3 - 3t^2 + 6$

(c) The particle's velocity is a quadratic expression:

$v = 3t^2 - 6t$

The particle is stationary when $v = 0$ so:

$3t^2 - 6t = 0$

$3t(t - 2) = 0$

$t = 0 \text{ or } t = 2\,\text{s}$

When the particle is stationary at $t = 2$ seconds, it changes direction and begins to move back towards its initial position. Next consider the time at which the particle returns to its initial position. This occurs when $s = 6$:

$s = t^3 - 3t^2 + 6$

When $s = 6$:

$6 = t^3 - 3t^2 + 6$

$t^3 - 3t^2 = 0$

$t^2(t - 3) = 0$

$t = 0 \text{ or } t = 3\,\text{s}$

Therefore, the particle returns to its initial position when $t = 3$.

When $t = 2$, $s = 2^3 - 3(2)^2 + 6 = 2\,\text{m}$

So the particle moves 4 metres during the first 2 seconds (from $s = 6$ to $s = 2$ m) and then another 4 metres to return to its starting position in the following 1 second.

In total, the particle travels 8 m in this time.

3. Particle P moves such that its velocity v m s^{-1} at any time t seconds can be calculated using the formula $v = -t^3 - 2t^2 + 3t \quad \text{for} \quad t \geq 0$

(a) Find the times at which P is at rest.

(b) Find the distance travelled by P in the first 2 seconds of its motion.

(c) Find the time at which P has its maximum velocity.

(a) P is at rest when $v = 0$:

$-t^3 - 2t^2 + 3t = 0$

$t^3 + 2t^2 - 3t = 0$

$t(t^2 + 2t - 3) = 0$

$t(t - 1)(t + 3) = 0$

$t = 0 \text{ or } t = 1\,\text{s}$

($t = -3$ can be rejected)

(b) $s = \int v\ \text{d}t$

$s = \int -t^3 - 2t^2 + 3t\ \text{d}t$

$s = -\frac{1}{4}t^4 - \frac{2}{3}t^3 + \frac{3}{2}t^2 + c$

> **Note:** No initial conditions are given, so the displacement can only be given in terms of c at any time. However, it is possible to work out the distance travelled between two times, as the magnitude of the difference between two displacements.

Calculate the displacement at the start and end times, as well as at the times the particle is stationary.

When $t = 0$, $s = c$

When $t = 1$, $s = \frac{7}{12} + c$

When $t = 2$, $s = -\frac{10}{3} + c$

Distance travelled between $t = 0$ and $t = 1$ is:

$\left|\left(\frac{7}{12} + c\right) - c\right| = \frac{7}{12}$

Distance travelled between $t = 1$ and $t = 2$ is:

$\left|\left(-\frac{10}{3} + c\right) - \left(\frac{7}{12} + c\right)\right| = \frac{47}{12}$

Total distance travelled in the first 2 seconds is:

$$\frac{7}{12} + \frac{47}{12} = 4.5\text{ m}$$

> **Note:** An alternative method is to carry out definite integration using limits of 0 and 1 to find the distance travelled between these times, then using limits of 1 and 2.

(c) The maximum velocity occurs when $\frac{dv}{dt} = 0$:

$$v = -t^3 - 2t^2 + 3t$$

$$\frac{dv}{dt} = -3t^2 - 4t + 3$$

When the velocity is at its maximum,

$$-3t^2 - 4t + 3 = 0$$

$$3t^2 + 4t - 3 = 0$$

$t = 0.535$ s or $t = -1.87$ s (which can be rejected)

$$\frac{d^2v}{dt^2} = -6t - 4$$

When $t = 0.535$, $\frac{d^2v}{dt^2} = -7.21$

$-7.21 < 0$ therefore the particle reaches its maximum velocity when $t = 0.535$ s.

You may have to differentiate and integrate a trigonometric function, an exponential function or a logarithmic function.

Worked Example

4. A particle's motion in a straight line is modelled such that its acceleration a m s^{-2} at time t seconds is given by $a = \sin 2\pi t$

The particle is initially at rest. Find:

(a) an expression for the velocity at time t seconds,

(b) the particle's minimum and maximum velocity,

(c) the distance travelled in the first 2 seconds.

(a) $v = \int \sin 2\pi t \ dt$

$$v = -\frac{1}{2\pi}\cos 2\pi t + c$$

The particle is initially at rest, so when $t = 0, v = 0$

$$\therefore 0 = -\frac{1}{2\pi}\cos(0) + c$$

$$c = \frac{1}{2\pi}$$

So:

$$v = -\frac{1}{2\pi}\cos 2\pi t + \frac{1}{2\pi}\text{ m s}^{-1}$$

(b) The minimum value of $-\frac{1}{2\pi}\cos 2\pi t$ occurs when $\cos 2\pi t = 1$. In this case:

$$v = -\frac{1}{2\pi}(1) + \frac{1}{2\pi}$$

$$= -\frac{1}{2\pi} + \frac{1}{2\pi}$$

$$= 0$$

The minimum velocity is 0 m s^{-1}.

The maximum value of $-\frac{1}{2\pi}\cos 2\pi t$ occurs when $\cos 2\pi t = -1$. In this case:

$$v = -\frac{1}{2\pi}(-1) + \frac{1}{2\pi}$$

$$= \frac{1}{2\pi} + \frac{1}{2\pi}$$

$$= \frac{1}{\pi}$$

The maximum velocity is $\frac{1}{\pi}$ m s^{-1}.

(c) Recall that the distance travelled is the area enclosed between the velocity-time graph and the time axis. Since the particle's minimum velocity is 0 m s^{-1}, the velocity-time graph never goes below the time axis. As such, it is safe to integrate the equation of the curve between 0 and 2 seconds.

$$s = \int_0^2 -\frac{1}{2\pi}\cos 2\pi t + \frac{1}{2\pi}\ dt$$

$$= \frac{1}{2\pi}\int_0^2 -\cos 2\pi t + 1\ dt$$

$$= \frac{1}{2\pi}\left[-\frac{1}{2\pi}\sin 2\pi t + t\right]_0^2$$

$$= \frac{1}{2\pi}[(0 + 2) - (0 + 0)]$$

$$= \frac{1}{\pi}\text{ or } 0.318\text{ m (3 s.f.)}$$

> **Note:** If parts of the curve were below the time axis it would be important to find each area above and below the axis separately.

Exercise 1B

1. A particle begins at rest at a fixed point O. Its acceleration is given by $a = 5t$, where $t \geq 0$ Find, in terms of t, expressions for:
   (a) the particle's velocity,
   (b) the particle's displacement from O.

2. At time $t = 0$ a particle P leaves the origin O and moves along the x-axis. At time t seconds the velocity of P is v m s^{-1}, where $v = 8t - t^2$
   (a) Find the maximum value of v.
   (b) Find the time taken for P to return to O.

3. A particle P moves along the x-axis. The acceleration of P at time t seconds, $t \geq 0$, is $(3t + 5)$ m s^{-2} in the positive x-direction. When $t = 0$, the velocity of P is 2 m s^{-1} in the positive x-direction. When $t = T$, the velocity of P is 6 m s^{-1} in the positive x-direction. Find the value of T.

4. A particle moves such that its displacement s metres from a fixed point O at time t seconds is given by $s = 6t^2 - 17t + 5$
   (a) Find the two times t_1 and t_2 at which the particle is at O.
   (b) Find the time at which the particle is at rest.
   (c) Find the distance travelled by the particle between t_1 and t_2.

5. A particle P moves such that its velocity v m s^{-1} at any time t seconds can be calculated using the formula $v = t^3 + 3t^2 - 10t$, for $t \geq 0$
   (a) Find the times at which P is at rest.
   (b) Find the distance travelled by P in the first 3 seconds of its motion.
   (c) Find the time at which P has its minimum velocity.

6. A particle P moves on the x-axis. At time t seconds, its acceleration is $(5 - 2t)$ m s^{-2}, measured in the positive x direction. When $t = 0$, its velocity is 6 m s^{-1} in the positive x direction. Find the time at which P is instantaneously at rest in the subsequent motion.

7. A particle moves in a straight line. At time t seconds after it begins its motion, the acceleration of the particle is $2\sqrt{t}$ m s^{-2} $(t \geq 0)$. After 1 second, the particle is moving with velocity $\frac{4}{3}$ m s^{-1}. Find the time taken for the particle to travel 10 m.

Exercise 1B...

8. A yo-yo moves vertically in a straight line so that at time t seconds its acceleration a m s^{-2} is $a = \frac{3}{5}(t - 1)$ m s^{-2} for $0 \leq t \leq 5$
   Initially the yo-yo is at rest.
   (a) Find the **speed** of the yo-yo after 1 second.
   (b) After how many seconds does the yo-yo again come to rest?
   (c) Find how far the yo-yo travels in the first 3 seconds.

9. A particle P moves along a straight line such that its velocity varies according to the formula $v(t) = t^4 - 8t^3 + 17t^2 - 4t$, where $0 \leq t \leq 4.5$ s
   (a) Given that $(t - 4)$ is a factor of $v(t)$, factorise $v(t)$ completely.
   (b) Find the times at which P is stationary.
   (c) Find an expression for the acceleration a in terms of t.
   (d) Given that $a = 0$ when $t = 2$, find the other two times at which the acceleration is 0.
   (e) Find the time at which P has its maximum velocity.

10. A rocket accelerates upwards with an acceleration of $\left(\frac{1}{4}t^2 + \frac{1}{8}t\right)$ m s^{-2}.
    (a) Find the time taken for the rocket's acceleration to reach 10 m s^{-2}.
    (b) Find the distance travelled by the rocket in this time. You may assume that the rocket starts from rest on its launchpad.
    (c) State any modelling assumptions made in the calculations.

1.3 Motion in Two Dimensions With Variable Acceleration

The variables involved in this section are:

s the displacement vector

v the velocity vector

a the acceleration vector

t time (a scalar)

The vector forms of the formulae listed in section 1.2 are as follows:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{s}}{dt^2} \quad \text{and} \quad \mathbf{v} = \int \mathbf{a}\,dt$$

$$\mathbf{v} = \frac{d\mathbf{s}}{dt} \quad \text{and} \quad \mathbf{s} = \int \mathbf{v}\,dt$$

Note: You may see **r** used as an alternative to **s** for the displacement vector.

Differentiation and integration of vectors

Using the above formulae involves differentiation and integration of vector quantities.

To do this, differentiate or integrate each component of the vector separately.

Worked Examples

5. The velocity of a particle at t seconds is given by:
$\mathbf{v} = 4t^3\mathbf{i} + 6t^2\mathbf{j}$
Given that the particle is initially at a fixed point O, find:
(a) an expression for the particle's displacement from O at time t,
(b) an expression for the particle's acceleration at time t.

(a) $\mathbf{s} = \int \mathbf{v}\,dt = \int 4t^3\mathbf{i} + 6t^2\mathbf{j}\,dt$

We can integrate each component separately. The vectors **i** and **j** are constants, so:

$\mathbf{s} = t^4\mathbf{i} + 2t^3\mathbf{j} + \mathbf{c}$

When $t = 0$, the particle is at O, so its displacement from O is $0\mathbf{i} + 0\mathbf{j}$. So:

$0\mathbf{i} + 0\mathbf{j} = (0)^4\mathbf{i} + 2(0)^3\mathbf{j} + \mathbf{c}$

$\mathbf{c} = 0\mathbf{i} + 0\mathbf{j}$

$\therefore \mathbf{s} = t^4\mathbf{i} + 2t^3\mathbf{j}$ m

Note: When integrating a vector, the constant of integration is also a vector quantity.

(b) Differentiate each component of the velocity vector separately:

$\mathbf{v} = 4t^3\mathbf{i} + 6t^2\mathbf{j}$

$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 12t^2\mathbf{i} + 12t\mathbf{j}$ m s^{-2}

Note: you can also write $\dot{\mathbf{v}}$ for $\frac{d\mathbf{v}}{dt}$

Note: Read the question carefully to work out whether you should give a distance or a displacement; a velocity or a speed; a vector acceleration or the magnitude of the acceleration.

6. A particle moves with a velocity $(4t\mathbf{i} + 6t^2\mathbf{j})$ m s^{-1}. Find the magnitude of the acceleration when $t = 2$

$\mathbf{v} = 4t\mathbf{i} + 6t^2\mathbf{j}$

$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 4\mathbf{i} + 12t\mathbf{j}$

When $t = 2$:

$\mathbf{a} = 4\mathbf{i} + 24\mathbf{j}$

Since we are asked for the magnitude of the acceleration:

$|\mathbf{a}| = \sqrt{4^2 + 24^2} = 4\sqrt{37} = 24.3$ m s^{-2} (3 s.f.)

If two particles collide, the **i** components of the two position vectors are equal and the **j** components of the position vectors are equal at the same time. The following example demonstrates how to find the time of collision.

Worked Example

7. The velocity of a particle P at time t seconds (where $t \geq 0$) is $\left((6t^2 - 16)\mathbf{i} + 15\mathbf{j}\right)$ m s^{-1}.
Initially P is at a point with position vector $(-2\mathbf{i} - 13\mathbf{j})$ m relative to a fixed origin O.
(a) Find the position vector of P after t seconds.
(b) A second particle Q moves with constant velocity $(16\mathbf{i} + 12\mathbf{j})$ m s^{-1}. Q is initially at a point whose position vector relative to O is $(-2\mathbf{i} - \mathbf{j})$ m. Show that P and Q collide.

(a) Let the position vector of P after t seconds be **p**.

$$\mathbf{p} = \int v\ dt = \int \left((6t^2 - 16)\mathbf{i} + 15\mathbf{j}\right)\ dt$$

$$\mathbf{p} = (2t^3 - 16t)\mathbf{i} + 15t\mathbf{j} + \mathbf{c}$$

When $t = 0$, $\mathbf{p} = -2\mathbf{i} - 13\mathbf{j}$

$\therefore -2\mathbf{i} - 13\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + \mathbf{c}$

$\mathbf{c} = -2\mathbf{i} - 13\mathbf{j}$

$\therefore \mathbf{p} = (2t^3 - 16t)\mathbf{i} + 15t\mathbf{j} - 2\mathbf{i} - 13\mathbf{j}$

$\mathbf{p} = \left((2t^3 - 16t - 2)\mathbf{i} + (15t - 13)\mathbf{j}\right)$ m

(b) Let the position vector of Q after t seconds be **q**. Since Q moves with constant velocity, we can use the formula:

$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$

where $\mathbf{r}_0$ is the initial position vector of the particle and **v** is the constant velocity.

So:

$\mathbf{q} = (-2\mathbf{i} - \mathbf{j}) + (16\mathbf{i} + 12\mathbf{j})t$

$\mathbf{q} = (16t - 2)\mathbf{i} + (12t - 1)\mathbf{j}$

If P and Q collide, the **i** components of their position vectors must be equal at the same time as their **j** components.

Find the time at which the coefficients of the **i** components are equal for P and Q:

$2t^3 - 16t - 2 = 16t - 2$

$2t^3 - 32t = 0$

$2t(t^2 - 16) = 0$

$t = 0$ seconds or $t = 4$ seconds.

These are the two times at which the two position vectors have the same **i** components.

For the coefficients of the **j** components:

$15t - 13 = 12t - 1$

$3t = 12$

$t = 4$

At time $t = 4$ seconds, the **i** components and the **j** components of the two position vectors are equal. This is the time at which the two particles collide.

The following example requires the times at which a particle is moving parallel to the vector **i**. This occurs when the **j** component of the **velocity vector** is zero. The second part asks for the times at which the particle is due east of the origin. This occurs when the **j** component of its **position vector** is zero.

Worked Example

8. A particle moves with an acceleration vector of

$\mathbf{a} = 12(t^2 - 1)\mathbf{j}$ m s^{-2}, $t \geq 0$

where **i** and **j** are unit vectors east and north of a fixed point O respectively. Initially, P is at the point with position vector $(2\mathbf{i} + 8\mathbf{j})$ m and has velocity $3\mathbf{i}$ m s^{-1}.

(a) Find the two times at which the particle is moving parallel to the vector **i**.

(b) Find the two times at which the particle is due east of O.

(a) $\mathbf{a} = 12(t^2 - 1)\mathbf{j}$

$\mathbf{a} = (12t^2 - 12)\mathbf{j}$

$\mathbf{v} = \int \mathbf{a}\,dt$

$\mathbf{v} = (4t^3 - 12t)\mathbf{j} + \mathbf{c}$

Initially, P has velocity $3\mathbf{i}$.

$\therefore 3\mathbf{i} = 0\mathbf{j} + \mathbf{c}$

$\mathbf{c} = 3\mathbf{i}$

$\mathbf{v} = 3\mathbf{i} + (4t^3 - 12t)\mathbf{j}$

When moving parallel to **i**, the **j** component of the velocity vector is zero.

$4t^3 - 12t = 0$

$4t(t^2 - 3) = 0$

$4t = 0$ or $t^2 = 3$

$t = 0$ or $t = 1.73$ s (3 s.f.)

(reject $t = -1.73$ s)

(b) $\mathbf{r} = \int \mathbf{v}\,dt = \int 3\mathbf{i} + (4t^3 - 12t)\mathbf{j}\,dt$

$\mathbf{r} = 3t\mathbf{i} + (t^4 - 6t^2)\mathbf{j} + \mathbf{c}$

Initially, P has position vector $2\mathbf{i} + 8\mathbf{j}$.

$\therefore 2\mathbf{i} + 8\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + \mathbf{c}$

$\mathbf{c} = 2\mathbf{i} + 8\mathbf{j}$

$\mathbf{r} = 3t\mathbf{i} + (t^4 - 6t^2)\mathbf{j} + 2\mathbf{i} + 8\mathbf{j}$

$\mathbf{r} = (3t + 2)\mathbf{i} + (t^4 - 6t^2 + 8)\mathbf{j}$

When the particle is due east of the origin, the **j** component of the position vector is zero.

$t^4 - 6t^2 + 8 = 0$

$(t^2 - 2)(t^2 - 4) = 0$

$t^2 = 2$ or $t^2 = 4$

$t = 1.41$ s (3 s.f.) or $t = 2$ s

(reject negative solutions)

Exercise 1C

1. A particle begins at rest at a fixed point O. Its acceleration vector **a** m s^{-2} is given by $\mathbf{a} = 12t(\mathbf{i} - \mathbf{j})$, where $t \geq 0$. Find, in terms of t, expressions for:

(a) the particle's velocity,

(b) the particle's displacement from O.

2. A particle P starts from rest at a fixed origin O. The acceleration of P at time t seconds (where $t \geq 0$) is $(6t\mathbf{i} + (5 - 3t^2)\mathbf{j})$ m s^{-2}

(a) Find the velocity of P when $t = 4$

(b) When $t = 2$

(i) find the position vector of P,

(ii) find the **distance** of P from O.

3. A particle moves such that its acceleration **a** m s^{-2} at time t seconds is given by:

$\mathbf{a} = 18t^2\mathbf{i} + 24t^2\mathbf{j}$

Given that the particle begins at rest at a fixed point O, find: