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Introduction

This book covers the revised specification for Unit AS 2: Applied Mathematics (Mechanics and Statistics) for CCEA, which was available for teaching from September 2018 onwards.

Accuracy

It is important to remember that all answers should be given either exact, or rounded to 3 significant figures. This advice is printed on the front page of all A Level Mathematics papers. Answer marks can be lost for rounding to any other level of accuracy.

Modelling

An important part of Applied Mathematics is **modelling**. Modelling questions may be set in relation to all topics in AS Applied Mathematics.

What does a modelling question look like?

A modelling question typically involves several of the following features, but not necessarily all of them:

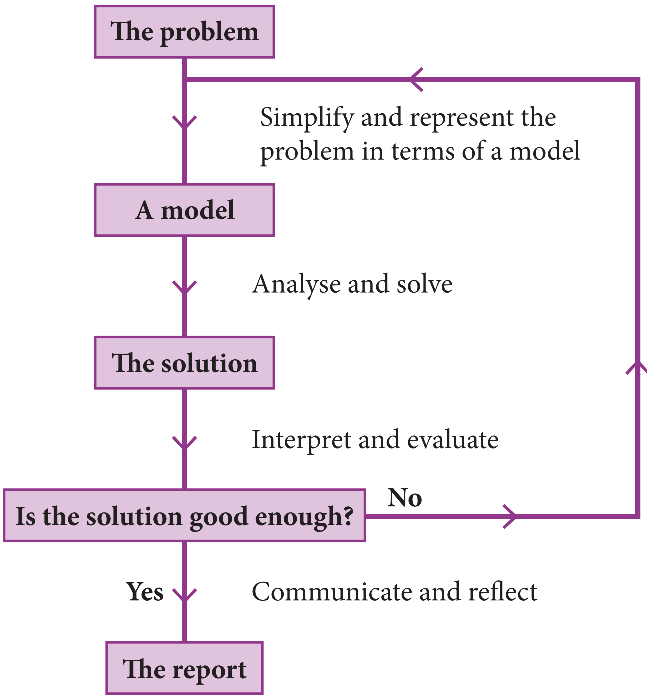
- There may be a requirement to make simplifications. The question will ask what simplifications or assumptions have been made.
- The candidate may be required to discuss the limitations of the model used.
- There may also be a requirement to refine or adapt the model or to consider different models.

The Modelling Cycle

The **Modelling Cycle** is outlined in the diagram. From the wording of the problem, the student should devise a way to model the situation. Simplifications and assumptions may be required.

The model should be applied to obtain a solution and this solution is interpreted and evaluated. At this point, it may become clear that certain assumptions were inappropriate, wrong, or not needed. It may be the case that different assumptions are required. In this way the model can be refined, and this modified version of the model is applied to the problem.

The final report should detail results, conclusions, any assumptions made and any limitations of the model being used. In AS Level Mathematics, the report will comprise the solution to the problem.



The Modelling Cycle

Chapter 1
Concepts in Mechanics

1.1 Introduction

This chapter is an introduction to some of the concepts used in AS Mechanics.

You will learn to use mathematical **models**. Models make assumptions to simplify a complex situation or process. You will learn when modelling assumptions are valid.

You will learn about the difference between **vectors** (such as velocity and force) and **scalars** (such as mass and time).

You will also learn about the SI units with which we measure many of these physical properties.

Key words

- **Model**: A mathematical representation of a real-life situation.
- **Vector**: A quantity with both size and direction, such as force or velocity.
- **Magnitude**: The length or size of a vector.
- **Resultant**: The sum of more than one vector.
- **Scalar**: A quantity that can be expressed as a single number. Mass, length and time are all scalar quantities.
- **Kilogram**: Unit of mass.
- **Metre**: Unit of length.
- **Second**: Unit of time.
- **Newton**: Unit of force.

Before you start

You should know:

- How to use vectors to represent translations.
- How to use Pythagoras' Theorem.
- How to use trigonometry in right-angled triangles.

What you will learn

In this chapter you will learn about:

- Modelling and modelling assumptions in Mechanics.
- Using vectors in Mechanics.
- The SI system of units.

In the real world...

In about 500 BC, the ancient Greeks were the first to present a mathematical model of the universe. Eudoxus' model placed a spherical Earth at the centre of the universe. The Sun, planets, and stars were then placed on giant transparent spheres surrounding it. A model of the universe that has the Earth at the centre is known as a geocentric model of the universe.

Aristotle extended Eudoxus' model of the universe in the 4th century BC. Aristotle's model of the universe was also geocentric, with the Sun, Moon, planets, and stars all orbiting the Earth inside Eudoxus' spheres. Aristotle believed the universe to be finite in space but to exist eternally in time.

Since then, astronomers have refined our model of the universe many times. Perhaps the most important change was an understanding that the Earth does not lie at the centre.

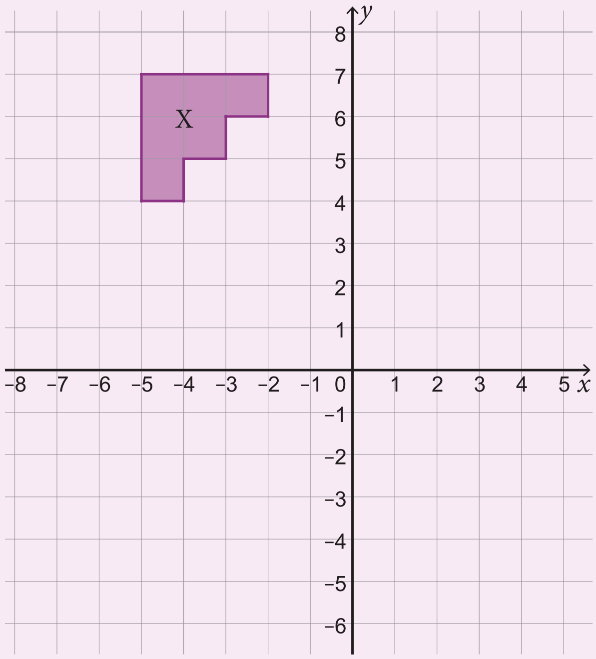
The concept of the Big Bang followed in the 20th century, the idea that all of matter exploded from a single point in space, and that the universe has been expanding ever since.

But the existing model of the universe is still incomplete. Observations suggest it is expanding more quickly than the model predicts. At the same time, the model suggests that the stars in galaxies should fly apart, but they do not. For these reasons, physicists have introduced the idea of dark matter and dark energy, but the exact nature of these concepts remains a mystery.

Many new ideas are being worked on, in an attempt to explain these anomalies, and others. One idea is that different rules of physics apply in different parts of the universe. Another, called string theory, is that the universe comprises sub-atomic strings oscillating in 11 dimensions. Of one thing we can be sure: our model of the universe will continue to evolve and develop as our understanding improves.

Exercise 1A (Revision)

1. Copy the diagram below. Draw shape X after each of the transformations given below. Label each of the images.

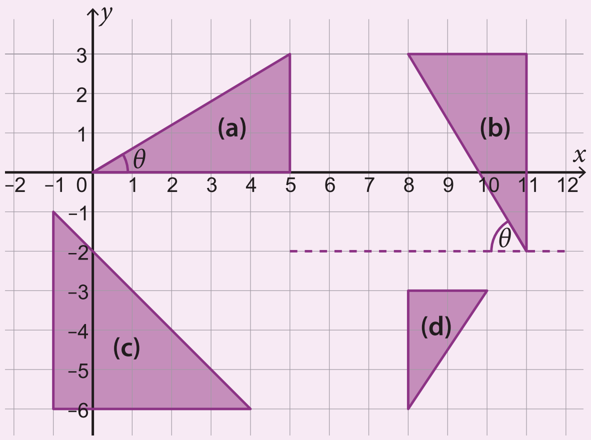


		Translation vector
(a)	X→Y	$\begin{pmatrix}3\\1\end{pmatrix}$
(b)	X→Z	$\begin{pmatrix}2\\-6\end{pmatrix}$
(c)	X→P	$\begin{pmatrix}-2\\-5\end{pmatrix}$
(d)	Z→Q	$\begin{pmatrix}-4\\-4\end{pmatrix}$
(e)	Z→R	$\begin{pmatrix}5\\1\end{pmatrix}$
(f)	R→S	$\begin{pmatrix}-2\\-4\end{pmatrix}$

Exercise 1A...

2. For each of the triangles (a) to (d) below:
   - (i) Find the length of the hypotenuse.
   - (ii) Find the angle between the hypotenuse and the horizontal. In the first two triangles the angle is marked.

   Round your answers to one decimal place where necessary.



1.2 Modelling

Modelling of a real-life situation is a key part of Mechanics. The modelling cycle was discussed in the introduction to this book.

Modelling assumptions

In all models, **assumptions** are made in order to simplify the problem and to allow you to use known mathematical technique to analyse real-life situations. You need to understand the significance of different modelling assumptions and how they affect the calculations in a particular problem.

Modelling assumptions can affect the validity of a model. For example, when modelling the landing of an aeroplane, it would not be appropriate to ignore the effects of the wind or the air resistance.

The following table shows some common modelling assumptions used in Mechanics.

Modelling assumption	What it means
An object is a **particle**	• The mass of the object is concentrated at a single point. • The object has no size, i.e. it is dimensionless. • Hence, air resistance can be ignored. • Rotational forces can also be ignored.
An object is a **rod**	• The object has a length, but no width or height, i.e. it is one-dimensional. • The mass is concentrated along a line. • The object is **rigid**, i.e. it does not bend or break.
An object is a **wire**	• A wire is a metal **rod**.
An object is **uniform**	• The mass of the object is distributed evenly. • This is equivalent to assuming that the weight force acts at a single point at the centre of the object, the centre of gravity.
An object is **light**	• The mass of the object is small when compared with other masses in the question. • The mass of the object can be neglected and will not appear in any equations. • The tension in a light string is the same throughout the string.
A string is **inextensible**	• The string does not stretch. • If two objects are connected by a taut, inextensible string, they will move with the same velocity and will have the same acceleration.
A surface is **smooth**	• There is no friction between the surface and any object in contact with it.
A surface is **rough**	• There is friction between the surface and any object in contact with it. • A friction force acts upon the object.
An object is a **bead**	• A bead is a particle that has a hole through it. • A bead is usually threaded **smoothly** onto a piece of string or a **wire**. • The tension in the string or wire is the same on either side of the bead.

Modelling assumption	What it means
A support is a **peg**	• A peg is a support. An object can rest on or be suspended from a peg. • A peg is dimensionless, i.e. it has no size. • It can be **rough** or **smooth**.
Air resistance	• The resistance force experienced by an object as it passes through the air. • Air resistance is usually assumed to be small compared with other forces, so is not included in the equations. • If an object is modelled as a **particle**, air resistance is always ignored, since a particle has no size.
Gravity	• Gravity is a force of attraction between all objects, but most importantly between an object and the Earth. • The acceleration due to the force of gravity is always vertically downwards. • The acceleration due to gravity is usually taken to be a constant 9.8 m s^{-2} and is denoted by g.

Worked Example

1. When a space capsule returns to Earth, parachutes are opened to allow the capsule to touch down, or splash down in the ocean, gently. This situation is to be modelled so that the time of touchdown can be estimated. Which of the following assumptions are valid?
   (a) The capsule is a particle.
   (b) The parachute is a particle.
   (c) The strings connecting the capsule to the parachute are light and inextensible.

(a) The capsule could be modelled as a particle. A particle has mass but no size; in this way, the effects of air resistance on the capsule would be ignored.
(b) It would not be appropriate to treat the parachute as a particle because the model should not neglect the effects of air resistance on the parachute.
(c) The strings can be considered light since their mass is negligible compared with the mass of the capsule. The amount they stretch will be small, so they can also be considered inextensible.

Exercise 1B

1. In modelling each of the following situations state at least one valid assumption that could be made.
   - (a) When a football is kicked by a goalkeeper, a model is used to work out its maximum height and its range (the distance travelled).
   - (b) An art teacher slides a pot of glue across a table towards a student. The model is to be used to calculate whether the pot stops on the table in front of the student, or whether it hits her.
2. Are the modelling assumptions listed valid when modelling each of the problems described?
   - (a) A pole vaulter attempts to set a new world record. A mathematical model is used to determine whether he clears the bar. Modelling assumptions used:
     - (i) The man is a particle.
     - (ii) His pole is a rod.
     - (iii) Gravity is a constant 9.8 m s^{-2}.
   - (b) A toboggan carrying two children slides down a snowy hill and onto a flat field, which is also covered in snow. A mathematical model is used to determine whether the toboggan crashes into a tree, which is standing in the field. Modelling assumptions used:
     - (i) The toboggan and the two children are modelled as a single particle.
     - (ii) The hill and the field are both smooth.
     - (iii) Gravity is a constant 9.8 m s^{-2}.
   - (c) A medical scientist is studying the effects of space flight on the human body. A mathematical model is used to determine the stresses placed on an astronaut's leg bones while she is orbiting the earth. Modelling assumptions used:
     - (i) An astronaut's leg bones are rods.
     - (ii) Gravity is a constant 9.8 m s^{-2}.

1.3 Quantities and Units in Mechanics

The International System of Units (the SI system) is the modern form of the metric system.

Fundamental quantities and units in the SI system: length, time, mass

The following base SI units are commonly used in Mechanics.

Quantity	Unit	Symbol
Mass	kilogram	kg
Length, distance and displacement	metre	m
Time	second	s

Warning: Weight and **mass** are not the same thing.

Mass is a measure of the amount of metal, wood, plastic, water, etc and is measured in **kilograms**.

Weight is a **force** acting on the object due to its mass and is measured in newtons (N).

For example, a mass of 2 kg experiences a downwards weight force of 19.6 N.

If this object is moved to the moon, its mass would remain as 2 kg, but the downwards weight force would be smaller because of the lower gravity on the moon.

Derived quantities and units: velocity, acceleration, force, weight

The table below shows some **derived** quantities. Derived units are compound units built from the base units.

Quantity	Unit	Symbol
Speed and velocity	metres per second	m s^{-1}
Acceleration	metres per second per second (or metres per second squared)	m s^{-2}
Forces, including weight	newton	N

Worked Example

2. Write the following quantities using SI units.

(a) 5.6 km **(b)** 25 g

(c) 900 km h^{-1} **(d)** 630 g cm^{-3}

(a) Multiply by 1000 to convert kilometres to metres:
5.6 km = 5600 m

(b) Divide by 1000 to convert grams to kilograms:
25 g = 0.025 kg

(c) To convert kilometres per hour to metres per second: multiply by 1000, then divide by 3600:
900 × 1000 ÷ 3600 = 250 m s^{-1}

(d) To convert grams per cubic centimetre to kilograms per cubic metre: divide by 1000, then multiply by 1 000 000:

$$\frac{630}{1000} \times 1\,000\,000 = 630\,000 \text{ kg m}^{-3}$$

Exercise 1C (Revision)

1. Write the following quantities using SI units.

(a) 30 km **(b)** 26 000 g

(c) 50 km h^{-1} **(d)** 25 g cm^{-3}

(e) 45 cm per minute **(f)** 120 g m^{-2}

(g) 0.97 km s^{-1} **(h)** 5.4×10^{-7} days

1.4 An Introduction to Vectors

In previous work you have used the quantities **distance**, **speed**, **time** and **mass.** These are all examples of **scalar** quantities. Scalars have a size, or **magnitude**, only.

In this book you will work with scalars, but also with **vectors**. The chapters on kinematics and forces both use vectors extensively.

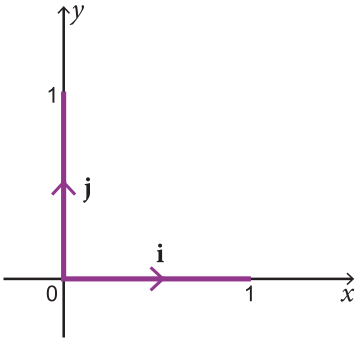
A vector is a quantity which has both magnitude and direction. The following quantities are vectors:

- **Displacement**: The distance travelled in a particular direction.
- **Velocity**: The speed in a particular direction.
- **Force**: Force is described both by its size and its direction.

In addition, the word **acceleration** can be used to refer to a scalar quantity (the rate of change of the speed) or to a vector quantity (the rate of change of the velocity).

In Pure Mathematics you may have learnt about **unit vectors**. A unit vector is any vector of length (or **magnitude**) 1.

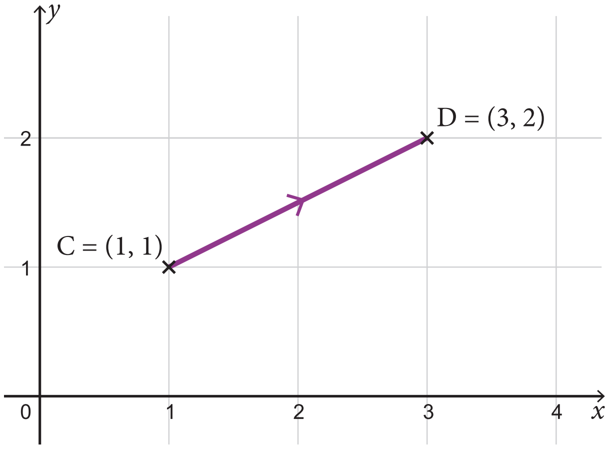
The special unit vectors **i** and **j** run parallel to the positive x and y axes respectively, as shown in the diagram.



Any two-dimensional vector can be built up using the unit vectors **i** and **j**.

Worked Example

3. The vector **p**, from the point C(1, 1) to the point D(3, 2), is shown below. Express this vector using the unit vectors **i** and **j**.



The vector **p** moves 2 units in the horizontal and 1 in the vertical:

$$\mathbf{p} = 2\mathbf{i} + \mathbf{j}$$

Alternatively, this can be written using column vector notation:

$$\mathbf{p} = \binom{2}{1}$$

Note: In your work you can use **i**-**j** notation or column vector notation. They are exactly equivalent. Both notations are used throughout this book.

Note: Vectors can have three components, although this is beyond the scope of the current A Level specification. Vectors with three components are used in the study of motion and forces in three-dimensional space. Vectors can also have more than three components in the abstract study of higher dimensions!

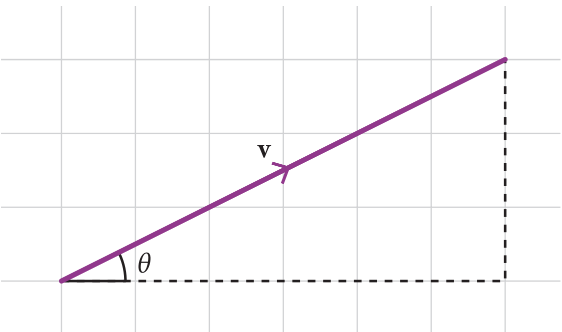
Worked Examples

4. The velocity of a particle is given by $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j}$ m s^{-1}. Find:

(a) The speed of the particle.

(b) The angle the direction of travel makes with the horizontal.

It is a good idea to sketch the vector:



A right-angled triangle has been formed.

(a) The speed is the magnitude (or size) of the velocity vector. This can be found using Pythagoras' Theorem:

$$|\mathbf{v}| = \sqrt{6^2 + 3^2} = 6.71 \text{ m s}^{-1}$$

(b) The angle that the vector makes with the horizontal can be found using trigonometry:

$$\tan\theta = \frac{3}{6}$$

$$\theta = 26.6°$$

Note: There are other ways this question could have been phrased. For example, it could have asked for the angle between the vector and the x-axis, or the angle between the vector $\mathbf{v}$ and the vector $\mathbf{i}$. Alternatively, it could have asked for the direction of the vector $\mathbf{v}$. If you are asked for the direction of a vector, find the angle between the vector and the positive x-axis, as in this example.

5. In the school playground, a girl walks from point A to point B, then to point C.

Her displacement from A to B is $\begin{pmatrix}5\\3\end{pmatrix}$ metres.

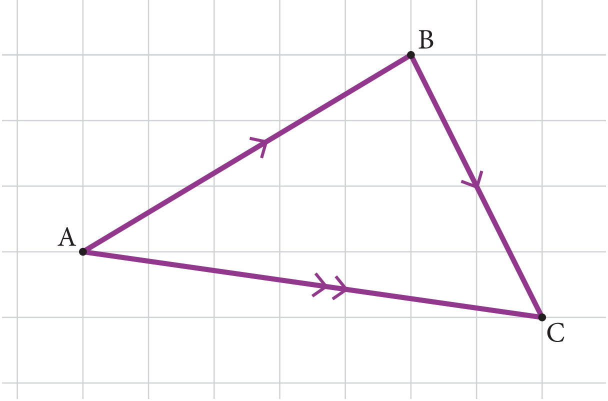
Her displacement from B to C is $\begin{pmatrix}2\\-4\end{pmatrix}$ metres.

(a) Find her displacement from A to C.

(b) Find the distance from A to C.

(c) Find the total distance the girl walks.

Sketch the girl's journey:



(a) Use the triangle law for vector addition:

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

Note: Vector $\overrightarrow{AC}$ is called the **resultant** of vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

$$\overrightarrow{AC} = \begin{pmatrix}5\\3\end{pmatrix} + \begin{pmatrix}2\\-4\end{pmatrix} = \begin{pmatrix}7\\-1\end{pmatrix}$$

Note: You can add vectors on your calculator to check your answer to part (a).

(b) The distance is the magnitude of the displacement vector:

$$\text{Distance} = |\overrightarrow{AC}| = \sqrt{7^2 + (-1)^2} = 7.07 \text{ m (3 s.f.)}$$

(c) The total distance the girl walks is the sum of the two distances AB and BC. Find the magnitudes of both vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$:

$$|\overrightarrow{AB}| = \sqrt{5^2 + 3^2} = 5.831 \text{ m}$$

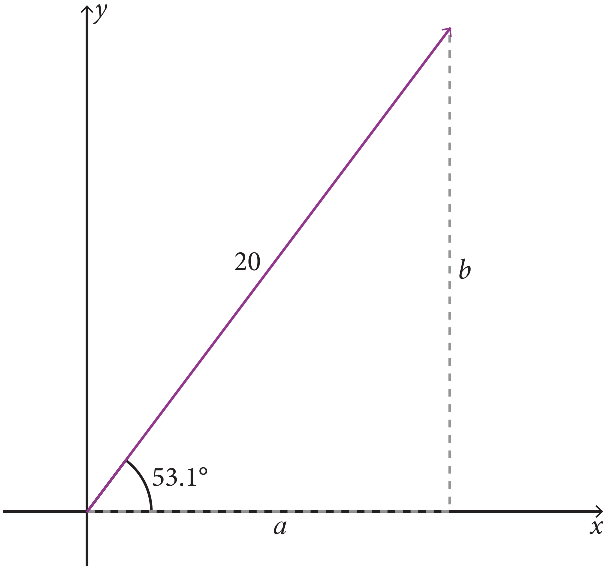
$$|\overrightarrow{BC}| = \sqrt{2^2 + (-4)^2} = 4.472 \text{ m}$$

$$\text{Total distance she walks} = 5.831 + 4.472 = 10.3 \text{ m (3 s.f.)}$$

You may be asked to find a vector in its component form, given the vector's magnitude and direction.

Worked Example

6. A vector has a magnitude of 20. The angle between the vector and the horizontal is 53.1°, as shown. Find the vector in its component form correct to 2 significant figures.



Using trigonometry in the right-angled triangle:

$$\cos 53.1° = \frac{a}{20}$$

$$\Rightarrow a = 20 \cos 53.1° = 12 \text{ (2 s.f.)}$$

$$\sin 53.1° = \frac{b}{20}$$

$$\Rightarrow b = 20 \sin 53.1° = 16 \text{ (2 s.f.)}$$

Therefore, the vector is $12\mathbf{i} + 16\mathbf{j}$

> **Note:** $-12\mathbf{i} - 16\mathbf{j}$ also has the required magnitude and angle, but it is clear from the diagram that both components of the vector are positive.

Exercise 1D

1. Are these quantities scalars or vectors?
- **(a)** Height
- **(b)** Mass
- **(c)** Speed
- **(d)** Time
- **(e)** Velocity
- **(f)** Distance from home
- **(g)** Displacement from a fixed point
- **(h)** Force

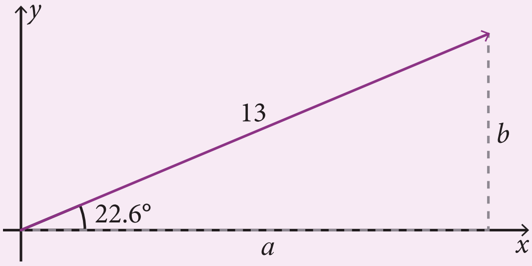
Exercise 1D...

2. Solve the following.
- **(a)** A train is travelling with a velocity of $-50\mathbf{i} + 120\mathbf{j}$ km h^{-1}. Find the speed of the train.
- **(b)** The acceleration of a particle is $-2\mathbf{i} + 7\mathbf{j}$ m s^{-2}. Find the magnitude of this acceleration.
- **(c)** A force of $-15\mathbf{i} - 20\mathbf{j}$ N is acting on a particle. Find the magnitude of the force.
- **(d)** A ship has a displacement vector of $30\mathbf{i} - 27\mathbf{j}$ km from its harbour H. Find the ship's distance from H.

3. A ball is thrown with velocity $\mathbf{v} = -8\mathbf{i} + 10\mathbf{j}$ m s^{-1}. Find:
- **(a)** the speed of the ball;
- **(b)** the angle that the direction of the ball's motion makes with the horizontal.

4. A boy cycles from point A to point B and then to point C. The displacement from A to B is $(-9\mathbf{i} + 4\mathbf{j})$ km. The displacement from B to C is $(5\mathbf{i} - 3\mathbf{j})$ km. Find:
- **(a)** the displacement of C from A;
- **(b)** the distance of C from A;
- **(c)** the total distance the boy cycles.

5. A vector has a magnitude of 13. The angle between the vector and the horizontal is 22.6°, as shown.



Find the vector in its component form, giving each component to the nearest integer.